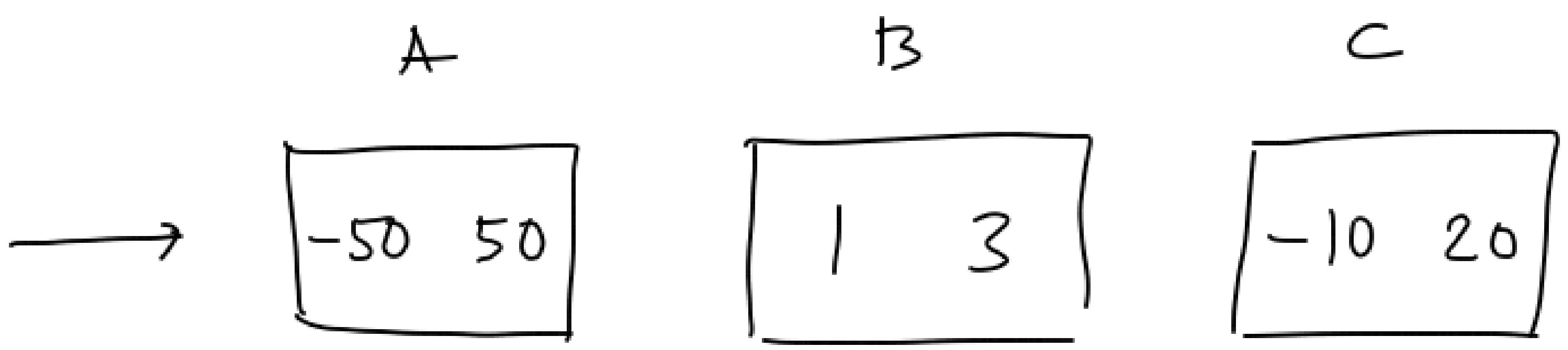
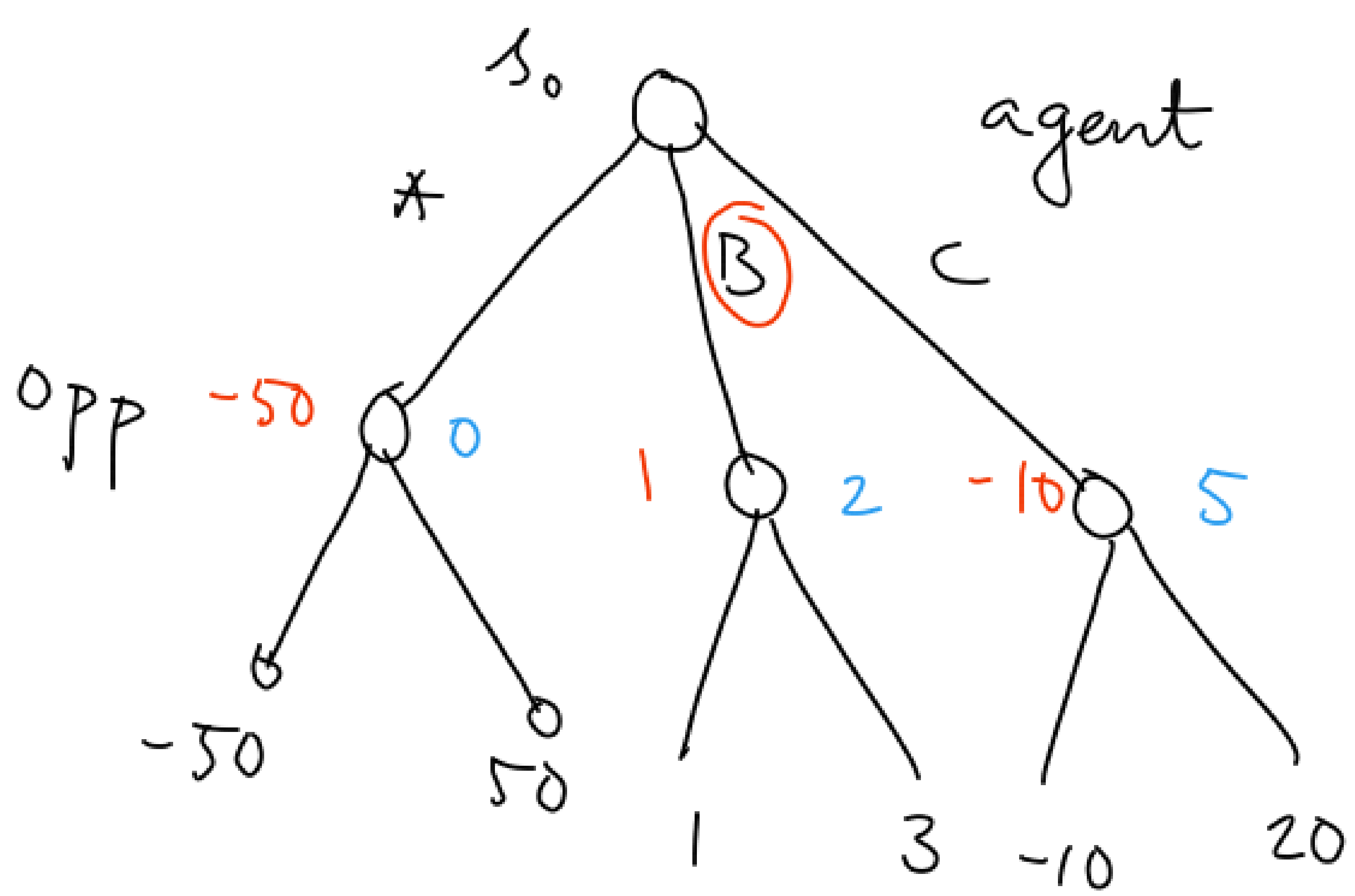
Lec 19: Two player competitive games



player 1 picks the bucket (agent)

player 2 picks a number (opp)
from the "selected" bucket.



opp: stochastic $\left(\frac{1}{2}, \frac{1}{2}\right)$

min player

Two player zero sum game (sequential move)

Players = { agent, opp }

s_0 = starting state

s is intermediate state:

- actions(s) = possible actions at state s
- player(s) = the player who makes the move at s
- succ(s, a) = resulting state if action a is taken at state s

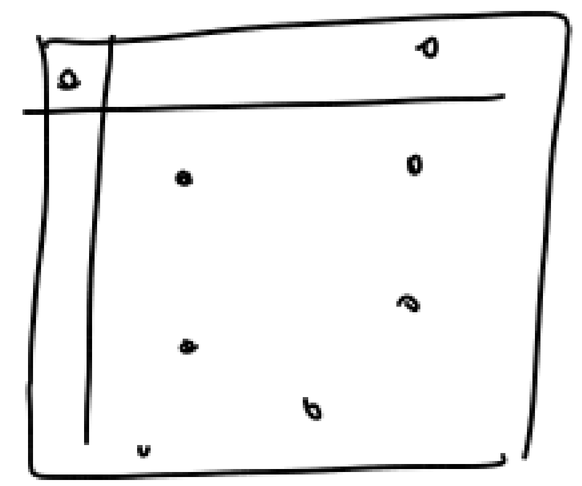
isEnd(s) = is state s an end state

utility(s) = agent's utility at an end state s.

E.g. Chess

players = {white, black}

s = a board position



actions(s) = all legal moves by the player(s)

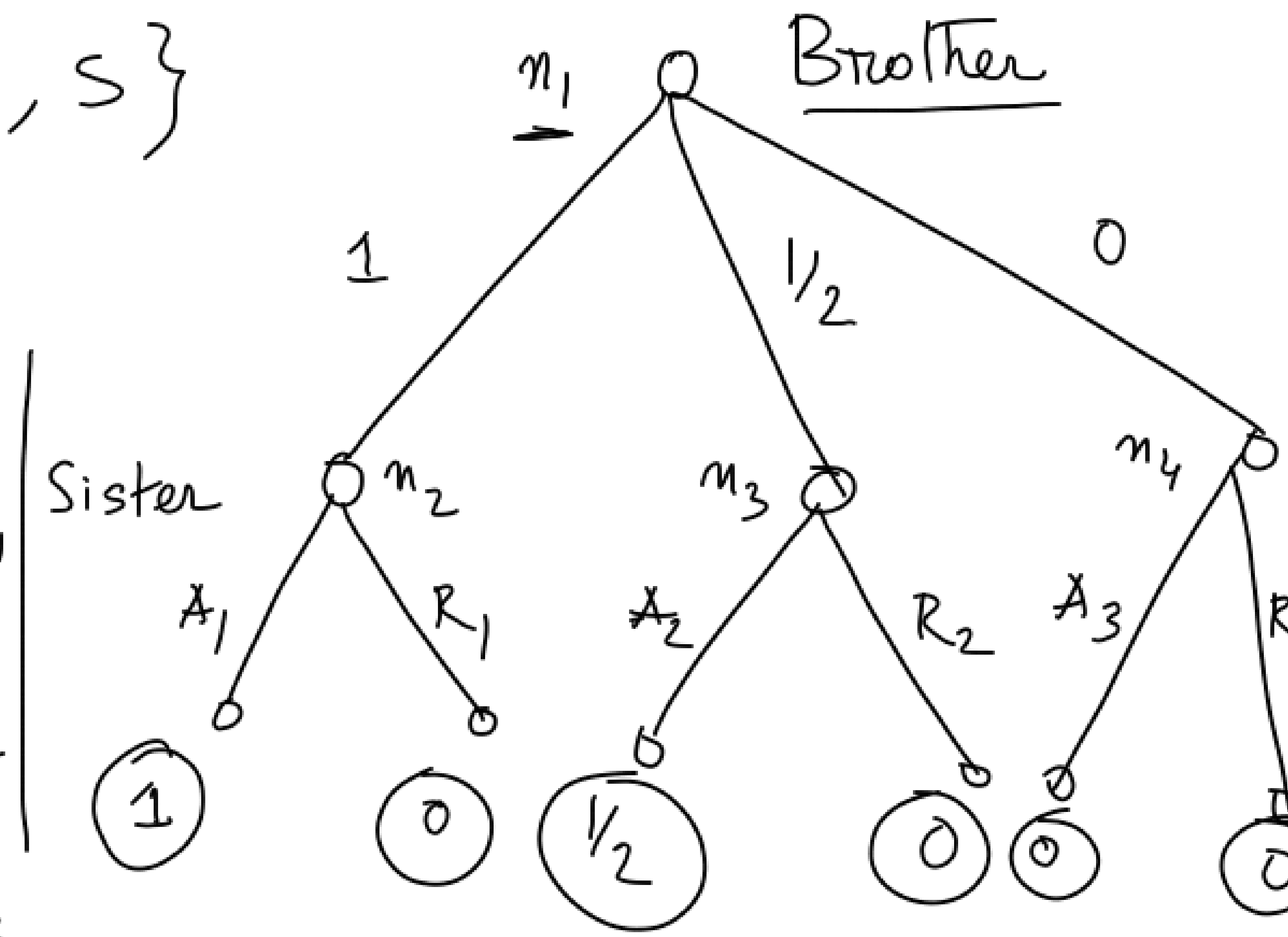
isEnd(s) = whether s is a checkmate or a draw

$$\text{utility}(s) = \begin{cases} +M & \text{if white wins} \\ -M & \text{if } \text{ " } \text{ loses} \\ 0 & \text{if draw} \end{cases}$$

Constant sum game

player = {B, S}

$$\text{actions}(s) = \begin{cases} \{1, \tfrac{1}{2}, 0\}, & s = n_1 \\ \{A_1, R_1\}, & s = n_2 \\ \{A_2, R_2\}, & s = n_3 \\ \{A_3, R_3\}, & s = n_4 \end{cases}$$



Strategy of a player

deterministic : $\pi_i(s) \in \text{actions}(s)$

if $\text{player}(s) = i$

randomized : $\pi_i(s) \in \Delta\, \text{actions}(s)$

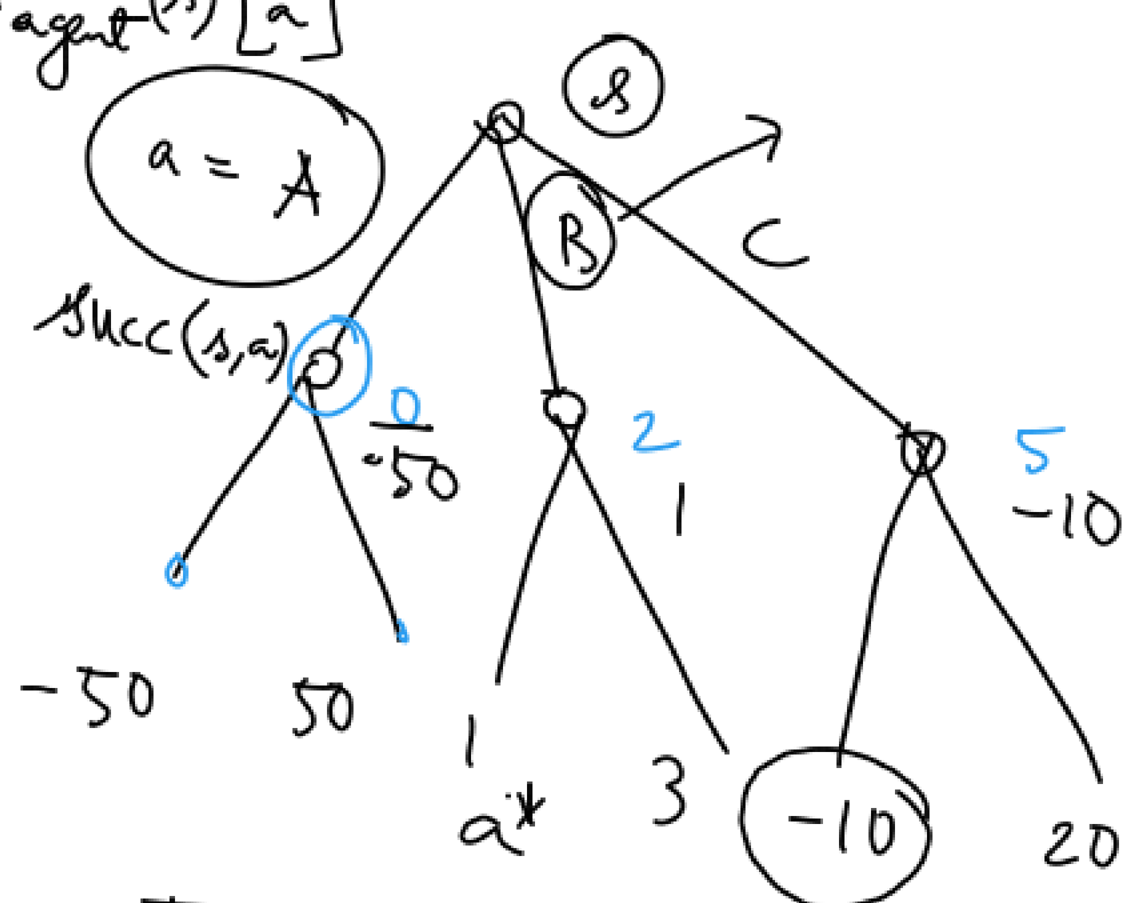
ΔA = set of all probability dist. over A

set

$\pi_B(n_1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$\left(\frac{2}{3}, \frac{1}{3}, 0\right)$

$\pi_{agent}(s)[a]$



$$a^* = \arg\max_{a_1 \in L_1} \boxed{\min_{a_2 \in L_2} u(a_1, a_2)}$$

$$\max_{a_1 \in L_1} \mathbb{E}_{a_2} u(a_1, a_2) \geq \mathbb{E}_{a_2} u(a_1^*, a_2)$$

Games with partial information

$$u_{agent}(s) = \begin{cases} \text{utility}(s) & \text{if isEnd}(s) \\ \sum_{a \in \text{actions}(s)} \underbrace{\cancel{\pi_{agent}(s)[a]}}_{\text{prob. that agent picks action } a} u_{agent}(\text{succ}(s,a)) & \text{if player}(s) = \text{agent} \\ \sum_{a \in \text{actions}(s)} \cancel{\pi_{opp}(s)[a]}\, u_{agent}(\text{succ}(s,a)) & \text{if player}(s) = \text{opp} \end{cases}$$

opp is stochastic

player is utility maximizer → max

$\max_{a \in \text{actions}(s)}$

$\min_{a \in \text{actions}(s)}$

Opp is utility minimizer for agent → min

$$U_{agent}(s) = \begin{cases} utility(s) & \text{if } isEnd(s) \\ \max_{a \in actions(s)} U_{agent}(succ(s,a)) & \text{if } player(s) = agent \\ \min_{a \in actions(s)} U_{agent}(succ(s,a)) & \text{if } player(s) = opp \end{cases}$$

$\pi_{agent}^{maxmin}(n_1) = B$

$\pi_{opp}^{min}(n_2)$ minimizing player's utility

Q: π_{agent}^{maxmin} is optimal $\pi_{opp}^{stochastic}$?

A: NO.

→ Equilibrium $\left(\pi_{agent}^{maxmin}, \pi_{opp}^{min}\right)$

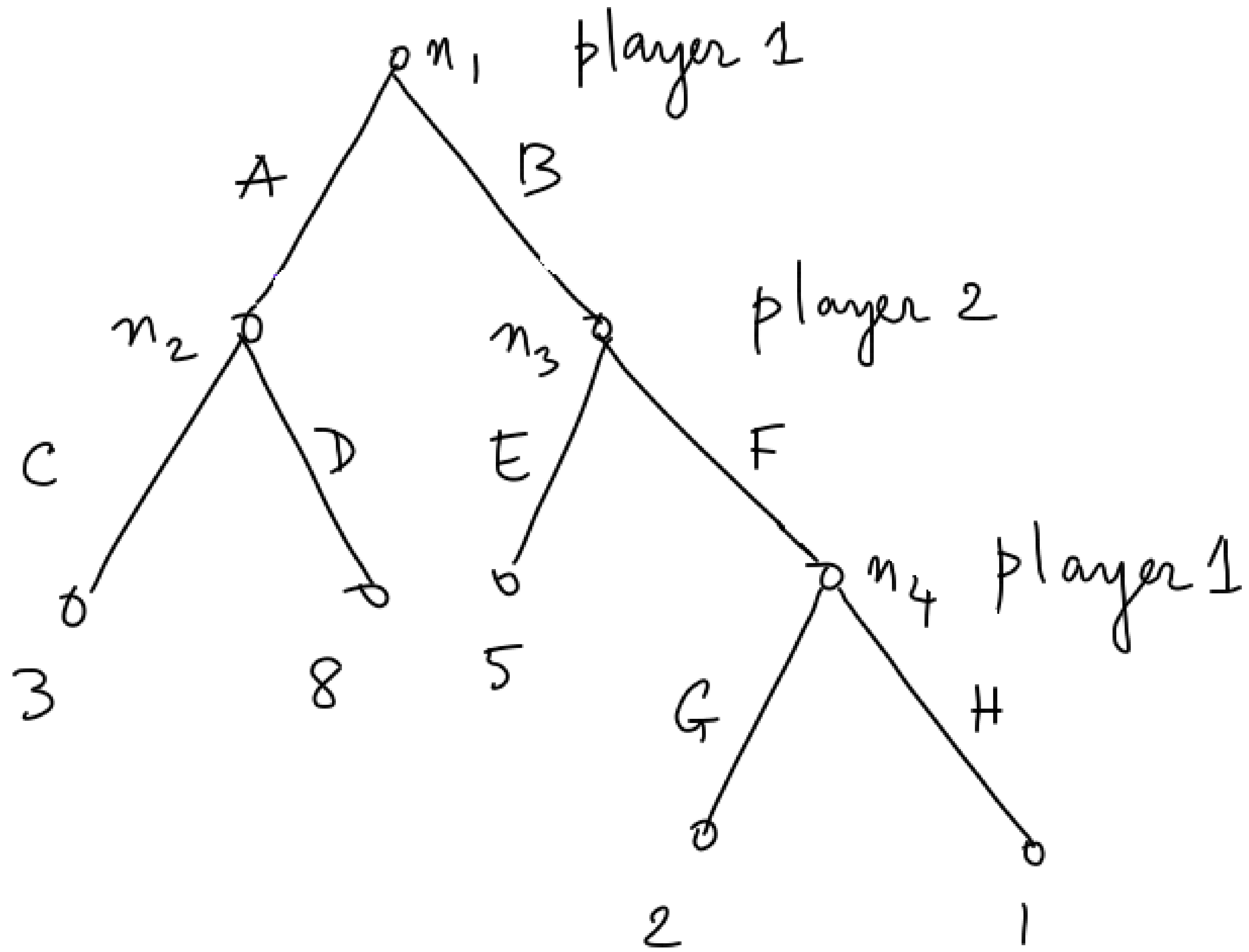
$\left(\pi_{agent}^{sto}, \underline{\pi_{opp}^{sto}}\right)$ → not equilibrium

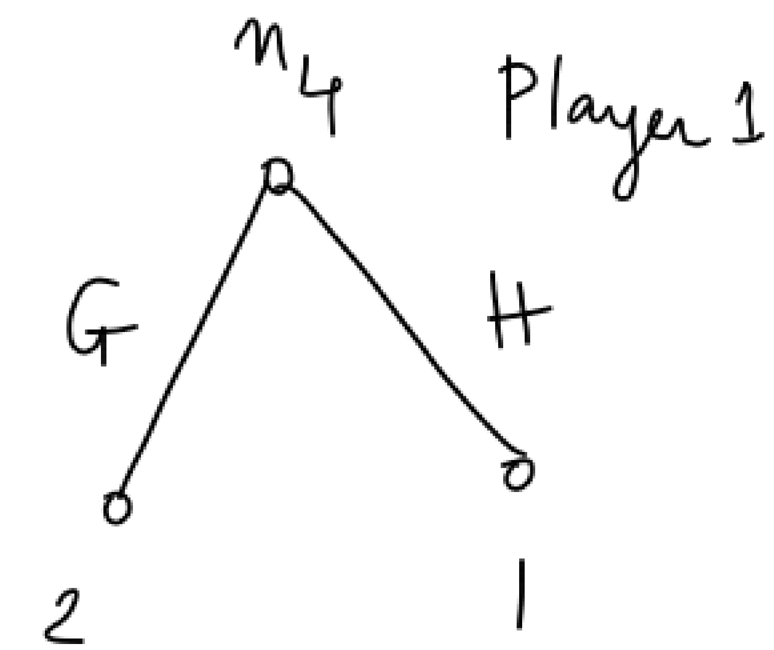
Subgame and Subgame Perfection

A subgame at s is the restriction of the game at the subtree rooted at s.

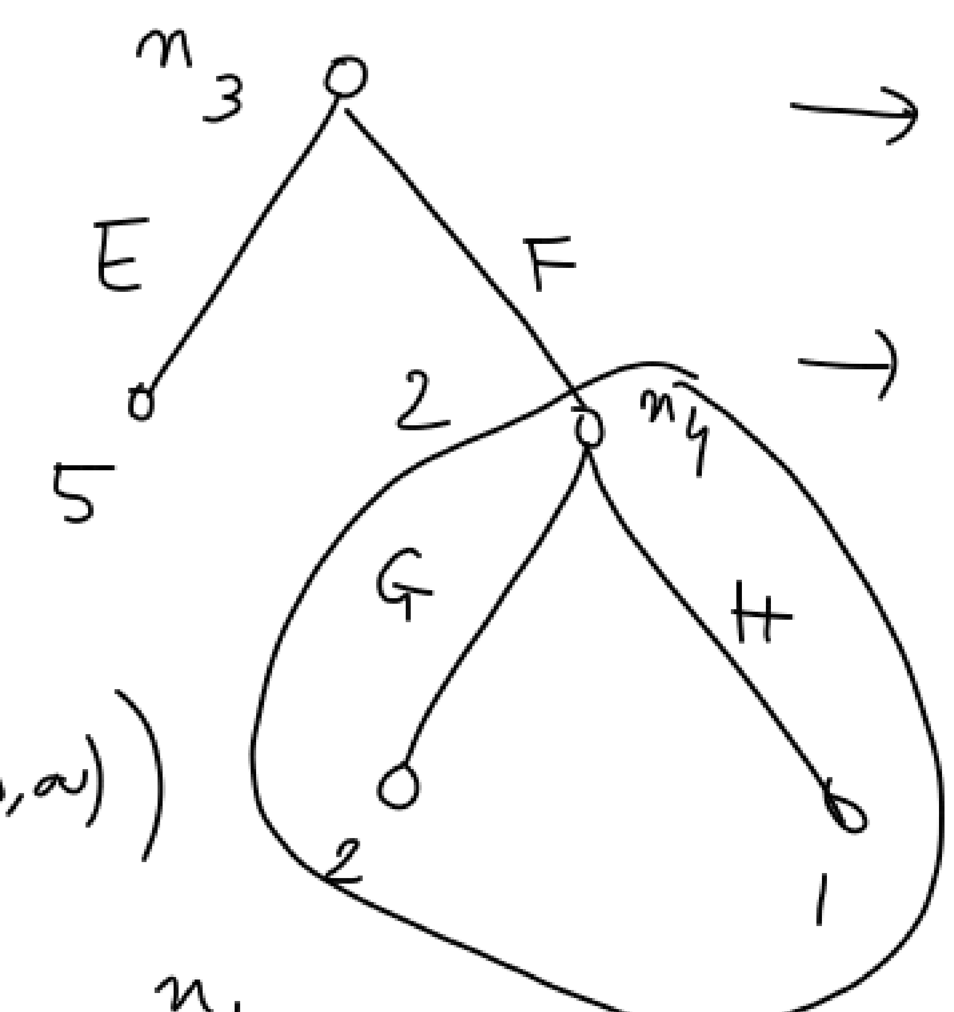
where isEnd(s) is false.

Subgame Perfect Equilibrium is an equilibrium at every subgame.



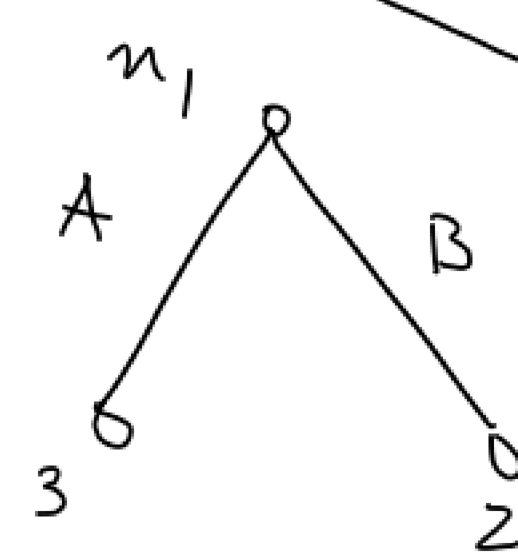


→ at n_4, player 1 should pick G



→ at n_3, player 2 should pick F

→ at n_2, " " " C



→ at n_1, player 1 should pick A.

player 1 is max → $\underset{a \in actions(s)}{\operatorname{argmax}} \; u_{agent}(succ(s,a))$

player 2 is min

Backward Induction:

```
function BackInd(s)
    if isEnd(s)
        return u_agent(s), φ
    if player(s) = agent then        opp
        bestUtil = -∞                +∞
        forall a ∈ actions(s) do
            utilAtChild, bestAvect ← BackInd(succ(s,a))
            if utilAtChild > bestUtil do        <
                bestUtil = utilAtChild
        bestAvect = append(a, bestAvect)
    return bestUtil, bestAvect.
```

Why play chess still?

Go, checker, TTT

Checkers game tree ~ 10^{20} nodes

Chess ~ 10^{40} nodes

Go ~ 10^{170} nodes

→ Solved in 2007 → 18 years of computation

it was a draw.

Speedup techniques