Lec 21 : Normal Form Game Representation

Arms Race

util$_2$

1 \ 2	s_{-i} Peace	War
s_i' Peace	5, 5	$\boxed{10}$, 10
s_i War	10, 0	1, 1

util$_1$

$$S_1 = \{P, W\} = S_2$$

$$u_1(P, W) = 0$$

$$u_2(P, W) = 10$$

$N = \{1, 2, \ldots, n\}$ set of players/agents

S_i = strategy set of player $i \in N$

s_i : one strategy of i , $s_i \in S_i$

$(s_1, s_2, \ldots s_i \ldots, s_n)$ = strategy profile

$= (s_i, s_{-i})$ $\qquad s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

$u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$

$u_i(s_i, s_{-i}) \in \mathbb{R}$ utility of player i

NFG Representation

$$\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

Dominated Strategy

A strategy s_i' of i is (strictly) dominated

if $\exists$ another strategy $s_i \in S_i$

s.t. $\forall s_{-i} \in S_{-i} = \times_{j \neq i} S_j$

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

dominated by s_i

s_i' is weakly dominated $\exists s_i \in S_i$

$\forall s_{-i} \in S_{-i}$ $\quad u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

$\exists \tilde{s}_{-i} \in S_{-i}$ $\quad u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$

Peace is strictly dominated in AR

① \ ② s_{-i}	D	$\boxed{E}$
A	5,5	0,5
$\boxed{B}$	5,0	1,1
C	4,0	1,1

A is WD

C is WD

E WDs D for pl 2

B WDs A, WDs C } pl 1

B WDS for player 1 in AR game

War is SDS for both

$$u_1(B,D) = u_1(A,D)$$
$$u_1(B,\tilde{E}) > u_1(A,E) \qquad (\tilde{s}_{-i})$$

$$u_1(B,\tilde{D}) > u_1(C,D) \qquad (\tilde{s}_{-i})$$
$$u_1(B,E) = u_1(C,E)$$

Dominant Strategy

A strategy that dominates every other strategy of that player

Strictly or Weakly Dominant strategy based on the type of domination

Equilibrium: If both players have SDSs / WDSs, the strategy profile of their SDS/WDS is called strictly/weakly dominant strategy equilibrium. (SDSE / WDSE)

AR Game: (War, War) is SDSE

Game 2: (B, E) is WDSE

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at least one player having a WDS while others have SDS $\Rightarrow$ WDSE

Professor's dilemma

~~games~~

P \ S	Listen	Sleep
Effort	100, 100	−10, 0
No effort	0, −10	0, 0

$(s_1^*, s_2^*) = (E, L)$

(s_1', s_2^*) (NE, L)

Pure strategy Nash Equilibrium

(Nash 1951)

" Some strategy profile from which unilateral (other players actions are fixed, only one player moves) deviations are not beneficial "

A PSNE is a strategy profile $(s_1^*, s_2^*, \dots, s_n^*)$ s.t.

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i \quad \forall i \in N$$

(E, L) is a PSNE

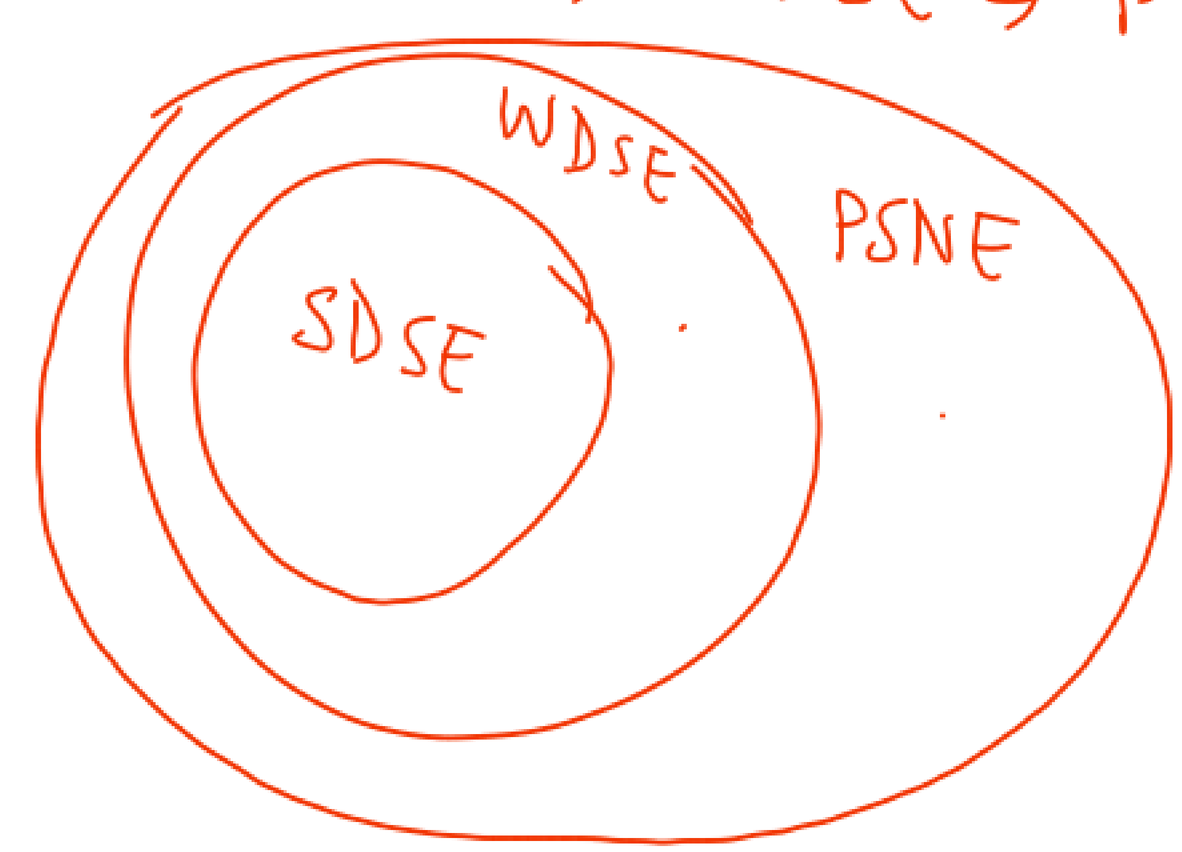
$$u_1(E, L) > u_1(NE, L)$$

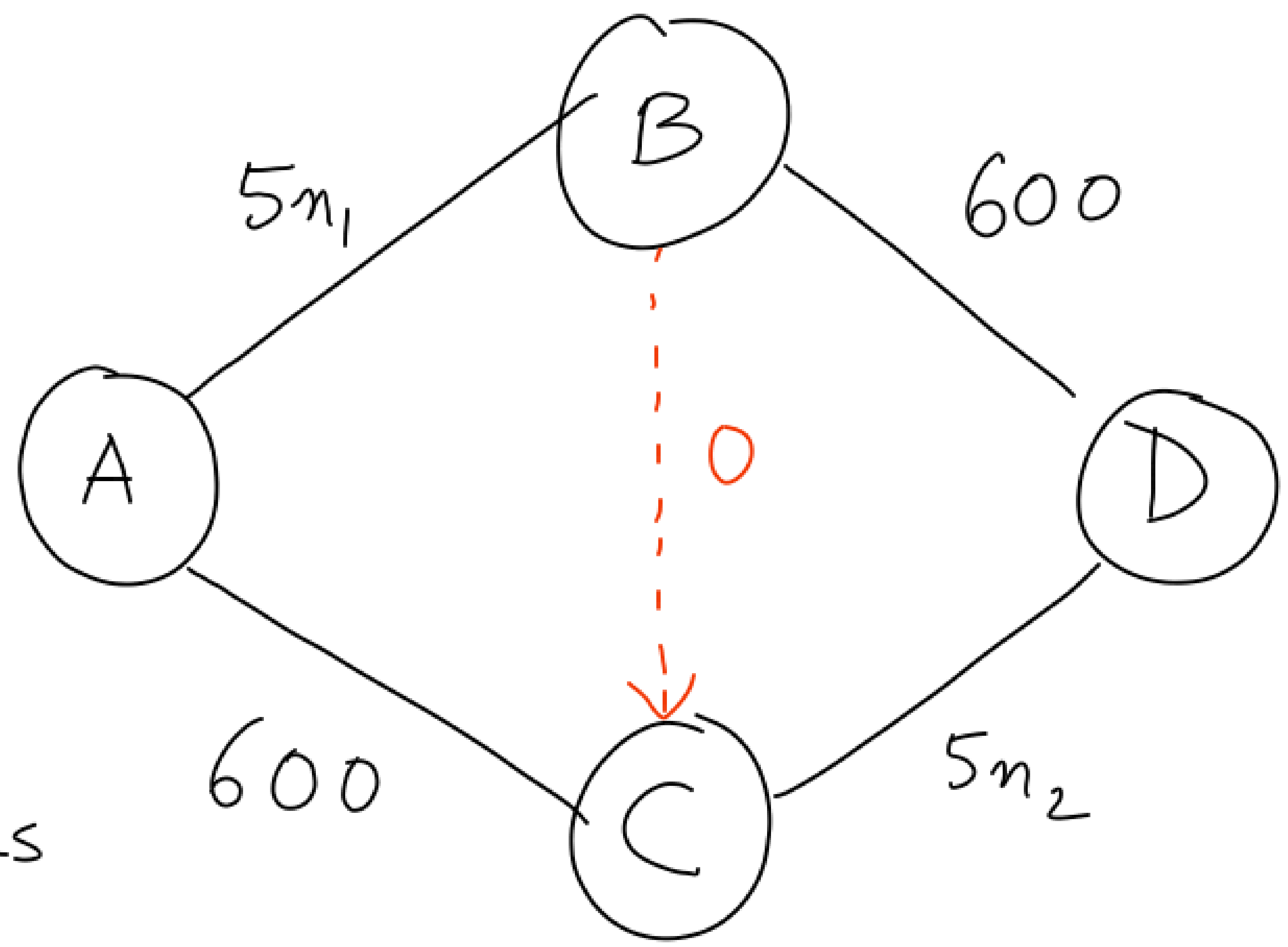
$$u_2(E, L) > u_2(E, S)$$

unilateral deviations

If (s_i^*, s_{-i}^*) is WDSE $\Rightarrow$ (s_i^*, s_{-i}^*) is a PSNE

SDSE $\Rightarrow$ WDSE $\Rightarrow$ PSNE





100 cycles

n_1, n_2 are the number of vehicles in that path

Equilibrium : (before)

50 cycles in each path (PSNE)

total time = 850

Equilibrium : (after)

100 cycles in ABCD (SDSE)

total time = 1000

Braess' Paradox

Recall: two player zero sum games

$$\overline{v} = \min_{s_2} \max_{s_1} u(s_1, s_2)$$

$$\underline{v} = \max_{s_1} \min_{s_2} u(s_1, s_2)$$

Fact: $\overline{v} \geq \underline{v}$

Theorem: A matrix game u has a PSNE (saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are the maxmin and the minmax strategies of players 1 and 2.

PSNE $\Leftrightarrow$ Saddle point

$$s_1^* := \arg\max_{s_1} \min_{s_2} u(s_1, s_2)$$

	L	C	R	min
L	-1	1	1	-1
C	1	-1	1	-1
R	1	1	-1	-1
max	1	1	1	

Penalty shootout game

			4/5	1/5
			1/2	1/2
			L	R
2/3	1/2	L	−1, 1	1, −1
1/3	1/2	R	1, −1	−1, 1

L L
L R
R L
R R

PSNE may not exist

mixed action $\left(\frac{4}{5}, \frac{1}{5}\right) = \sigma_2$

= the player picks L w.p. $\frac{4}{5}$ and R w.p. $\frac{1}{5}$

A mixed strategy is a probability distribution over the pure strategies

$$u_1(L,L) \times \frac{4}{5} + u_1(L,R) \times \frac{1}{5} = u_1(L, \sigma_2)$$

$\sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$

$$u_1(\sigma_1, \sigma_2) = u_1(L,L) \times \frac{2}{3} \times \frac{4}{5} + u_1(LR) \times \frac{2}{3} \times \frac{1}{5} + u_1(RL) \times \frac{1}{3} \times \frac{4}{5} + u_1(RR) \times \frac{1}{3} \times \frac{1}{5}$$

$$u_1\left(L, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = -3/5$$

$$u_1\left(R, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = 3/5$$

$$\sigma_2' = \left(\frac{1}{5}, \frac{4}{5}\right)$$

$$u_1\left(L, \left(\frac{1}{5}, \frac{4}{5}\right)\right) = 3/5$$

R . . . $= -3/5$

$$u_1\left(L, \left(\frac{1}{2}, \frac{1}{2}\right)\right) = 0$$

$$u_1\left(R, \left(\frac{1}{2}, \frac{1}{2}\right)\right) = 0$$

$$u_1\left(\left(\tfrac{1}{2},\tfrac{1}{2}\right),\underline{\left(\tfrac{1}{2},\tfrac{1}{2}\right)}\right) \geq u_1\left(\sigma_1,\underline{\left(\tfrac{1}{2},\tfrac{1}{2}\right)}\right)$$

A strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a mixed strategy NE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i' \quad \forall i \in N.$$