Lec 22: Simultaneous Move Games

$\frac{1}{5}$ $\frac{4}{5}$ $\frac{4}{5}$ $\frac{1}{5}$ $= \sigma_2$

	$\frac{1}{2}$ L	$\frac{1}{2}$ R
$\frac{2}{3}$ $\frac{1}{2}$ L	$-1, 1$	$1, -1$
$\frac{1}{3}$ $\frac{1}{2}$ R	$1, -1$	$-1, 1$
	$\frac{1}{2}$	$\frac{1}{2}$

utility of player 1 is the same for both L and R if player 2 chooses $\sigma_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$

player 2 is also indifferent between L & R if player 1 chooses $\sigma_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$u_1(\sigma_1, \sigma_2) = \sum_{s_2 \in S_2} \sum_{s_1 \in S_1} \sigma_1(s_1)\, \sigma_2(s_2)\, u_1(s_1, s_2)$$

$$u_1\left(\left(\tfrac{2}{3}, \tfrac{1}{3}\right), \left(\tfrac{4}{5}, \tfrac{1}{5}\right)\right) = \frac{2}{3} \times \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1)$$

$$+ \frac{1}{3} \times \frac{4}{5} \cdot (1) + \frac{1}{3} \cdot \frac{1}{5} (-1)$$

expected utility of player i

$$\boxed{u_i(\sigma_i, \sigma_{-i})} = \sum_{s_n \in S_n} \cdots \sum_{s_1 \in S_1} \sigma_1(s_1) \cdots \sigma_n(s_n)\, u_1(s_1, s_2, \ldots, s_n)$$

e.g.

$$\underbrace{\left(\tfrac{2}{3}, \tfrac{1}{3}\right)}_{\sigma_1}, \underbrace{\left[\left(\tfrac{4}{5}, \tfrac{1}{5}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right]}_{\sigma_{-1}}$$

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i' \quad \forall i \in N$$

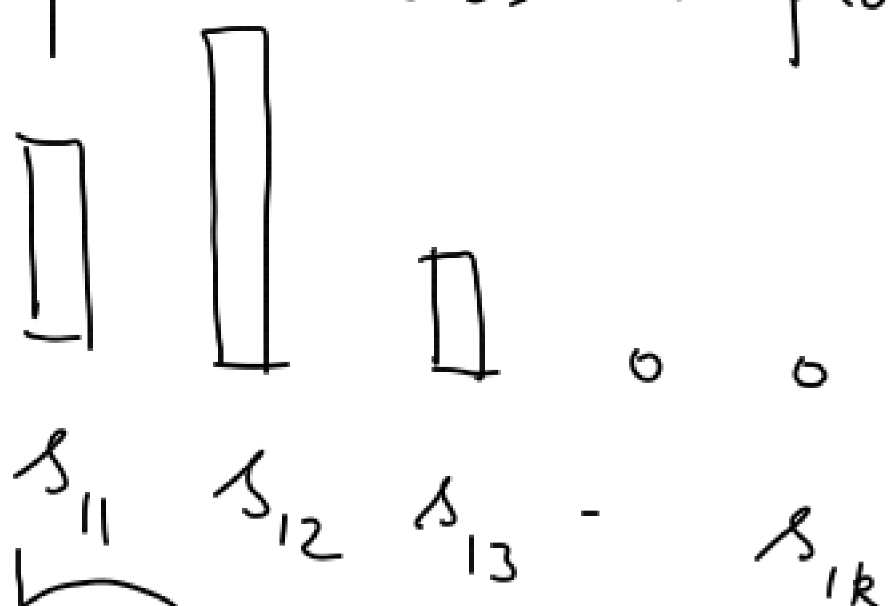
$$\left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)$$

How to find MSNEs?

"Support" of a probability distribution σ

: subset of $\underbrace{\text{state space}}_{\text{pure strategies}}$ where positive prob. mass is placed by σ

σ_1



$\underbrace{s_{11} \; s_{12} \; s_{13}}_{\delta(\sigma_1)}$

$$u_i(s_i, \sigma_{-i}^*) = \sum_{\substack{s_j \in S_j \\ j \neq i}} \prod_{j \neq i} \sigma_j(s_j) \, u_i(s_i, s_{-i})$$

Theorem: A mixed strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is an MSNE if and only if $\forall i \in N$

(1) $u_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in \underbrace{\delta(\sigma_i^*)}_{\text{support of } \sigma_i^*}$.

(2) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$

$s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$.

	L	R
→ L	-1, 1	1, -1
R	1, -1	-1, 1

Possible supports : $\{L\}$, $\{R\}$, $\{L, R\}$

1 $(\{L\}, \{L\}) \longrightarrow \times$ $u_1(L, L) < u_1(R, L)$ violates 2

δ_1 δ_2

2 → same

3 $(\{L\}, \{L, R\})$ $\times$ $u_2(L, L) \neq u_2(L, R)$

4 $(\{L, R\}, \{L, R\})$

L R

$\sigma_1 = (p, 1-p)$

$\sigma_2 = (q, 1-q)$

Cond 1 for player 1

$$u_1(L, (q, 1-q)) = u_1(R, (q, 1-q))$$

$$\Rightarrow q \cdot (-1) + (1-q) \cdot 1 = q(1) + (1-q)(-1)$$

$$\Rightarrow q = \frac{1}{2}$$

$$u_2((p, 1-p), L) = u_2((p, 1-p), R)$$

$$\Rightarrow p = 1/2$$

Prof's dilemma → find the MSNE.

So far

"Given a game, what is the rational outcome?"

How about the question

"Given an outcome, how game should be designed?"

s.t. in the equilibrium of that game, the desired outcome will be obtained.

Mechanism Design / Social Choice

Voting:

$N = \{1, \dots, n\}$

$A = \{a_1, a_2, \dots, a_m\}$

set of alternatives

Every agent has strict preferences over A

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{pmatrix} = P_i$$

$$\times \begin{pmatrix} & a_1 & \\ a_2 & & a_3 \end{pmatrix}$$

Voting / Social Choice function

$$f(P_1, P_2, \dots, P_n) \in A$$

e.g.

$$f\left(\begin{array}{cccc} \underline{P_1} & \underline{P_2} & \underline{P_3} & \cdots \\ a_2 & a_1 & a_4 & \cdots \\ a_1 & a_2 & \vdots & \\ a_4 & a_3 & \vdots & \\ a_3 & a_4 & \vdots & \end{array}\right) = a_4$$

Common Voting Rules

Each voter votes for one candidate (most preferred). The candidate with most votes win. (Plurality)

	1	2	3	4	5		Plurality
w_1	$\cancel{a}^3$	$\cancel{a}^3$	$\cancel{b}^1$	c	$\cancel{d}^2$	3	winner
w_2	$\cancel{b}^1$	$\cancel{b}^1$	c	$\cancel{b}^1$	$\cancel{b}^1$	2	a
w_3	$\cancel{d}^2$	c	$\cancel{d}^2$	$\cancel{d}^2$	c	1	
$\vdots$	c	$\cancel{d}^2$	$\cancel{a}^3$	$\cancel{a}^3$	$\cancel{a}^3$	0	
w_m							

$w_i \geq w_{i+1}$

Scoring rule votes.

Borda

$w_1 = m-1$

$w_2 = m-2$

$\vdots$

$w_{m-1} = 1$

$w_m = 0$

Borda voting rule

a : 6
b : 11
c : 7
d : 6

Single Transferable Vote (STV)

Runs in multiple rounds, in each round one candidate with minimum plurality score is eliminated.

sequential elimination
→ c in the given example.

Condorcet Consistency

A condorcet winner is a candidate who beats every other candidate in pairwise elections.

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$a \leftrightarrow b \rightarrow a$
$b \leftrightarrow c \rightarrow b$
$c \leftrightarrow a \rightarrow c$

A condorcet consistent voting rule always outputs a condorcet winner if it exist.

Is plu CC?

a	b	c
b	a	a
c	c	b
30%	30%	40%

pairwise $a \leftrightarrow b$ $\overset{a}{70-30}$

$a \leftrightarrow c$ $\underset{a}{60-40}$

plurality $\rightarrow$ c

not condorcet consistent.

Copeland rule:

Copeland scores for a candidate (a)

= # of wins it has pairwise

Candidate with highest Copeland score wins.

Copeland rule is Condorcet Consistent.