Lec 23: Voting and Stable Matching

Voting rules

1. Plurality
2. Borda's rule
3. STV

Condorcet Consistency

if a candidate beats all other candidates in pairwise election, that candidate should win.

4. Copeland voting

Q: What other desirable properties?

Unanimity:

A voting rule f is UN if for every profile P s.t. $P_1(1) = P_2(1) = \cdots = P_n(1) = a$ (say) it holds that $f(P) = a$.

$$P_i = \begin{pmatrix} a \\ b \\ c \\ \vdots \end{pmatrix} \begin{matrix} \leftarrow P_i(1) \\ \leftarrow P_i(2) \\ \vdots \\ \leftarrow P_i(m) \end{matrix}$$

$$f(\underbrace{P_1, P_2, \ldots, P_n}_{P}) \in A$$

Manipulable

A voting rule f is *manipulable* if

$\exists\, i \in N$, profile $P = (P_i, P_{-i})$

s.t.

$f(P_i', P_{-i})$ is more preferred by i under P_i than $f(P_i, P_{-i})$ for some P_i'

A voting rule f is non-manipulable if f does not satisfy the above.

$f(P_i, P_{-i}) = b$

$f(P_i', P_{-i}) = a$

P_i	P_i'	P_{-i}
⋮	⋮	
a	b	
⋮	a	
b	⋮	
⋮	⋮	

Plurality

	1	2	3	4	5	6	P_5'
3	a	a	b	b	c	d	b
2	b	b	a	a	b	a	c
1	c	d	c	c	a	b	a
0	d	c	d	d	d	c	d

Borda's rule

$f_{BORDA}(P) = a$

P_5'
b
c
d
a

a → 13

b → 13

c → 6

d → 4

STV: HW

Copeland

a	b	c
b	c	a
c	a	b

Score
$a \to 1$
$b \to 1$
$c \to 1$

P_2'
c
b
a

$a \to 1$
$b \to 0$
$c \to 2$

WLOG say a is the Copeland winner

$f(P) = a$

$f(P_2', P_{-2}) = c$

Dictatorial

$\exists\, d \in N$ s.t. $f(P_d, P_{-d}) = P_d(1)$

DICT is UN & non-manipulable.

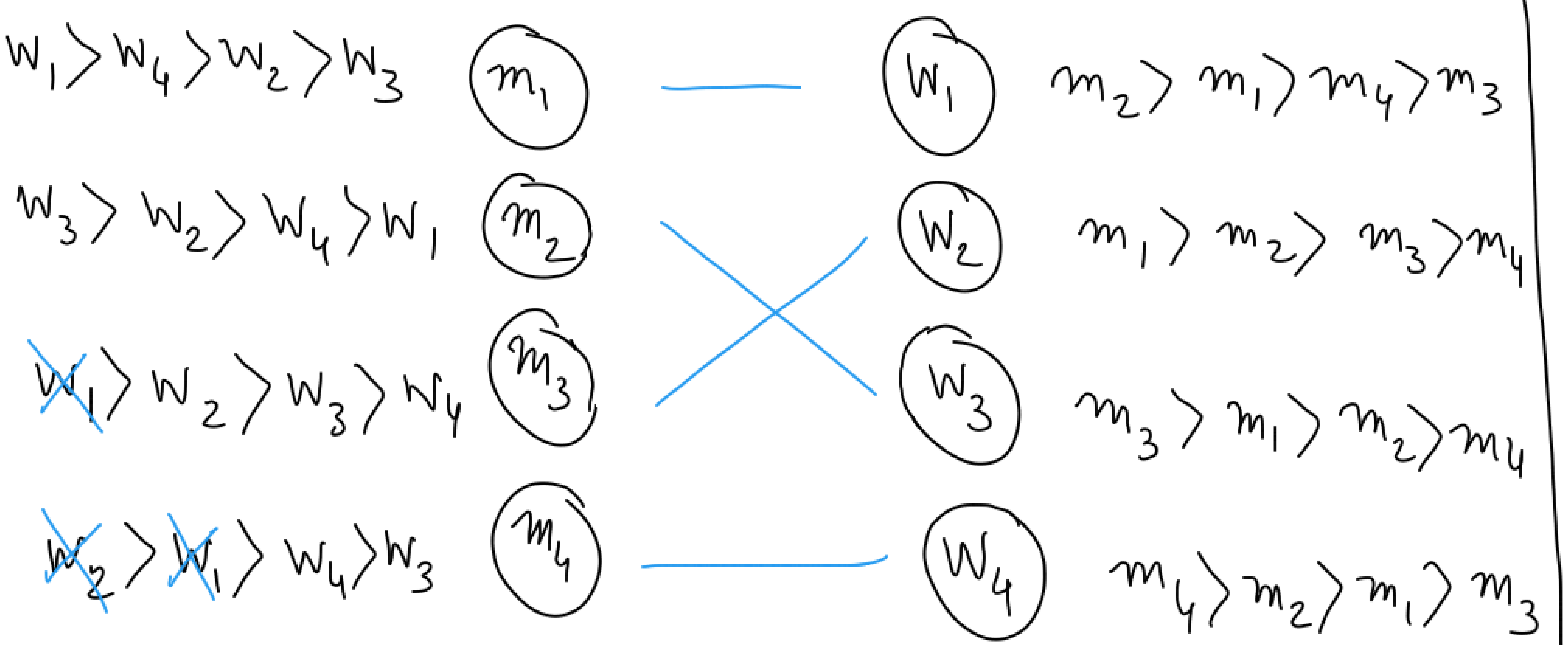
Gibbard-Satterthwaite:

If voters can have all possible strict preferences over the candidates and $|A| \geq 3$, then every UN and non-manipulable voting rule is a dictatorship.

Stable Matching

How to find a stable matching algorithmically?

Gale-Shapley Deferred Acceptance



men-proposing version

Round 1: Each unmatched man proposes to the most preferred woman who hasn't rejected him

$m_1 \to w_1$, $m_2 \to w_3$, $m_3 \to w_1$, $m_4 \to w_2$

Each woman tentatively accepts the most preferred man from the existing proposals.

$m_1 - w_1$, $m_2 - w_3$, m_3, $m_4 - w_2$

R2: $m_3 \to w_2$, $m_3 - w_2$, m_4

R3: $m_4 \to w_1$, m_4

R4: $m_4 \to w_4$, $m_4 - w_4$

function mpStableMatching:

M: set of men
W: set of women
$\Big\} \; |M| = |W| = n$

initialize all $m \in M$, $w \in W$ as free

while $\exists\, m$ who is free

m proposes w who is most preferred by m and has not rejected m

if w is free

(m, w) is tentative match

if some (m', w) already exists

if w prefers m over m'

(m, w) tentative match

m' free

else

m free

all tentative are final.

Q: Does it converge?

Does it give a stable match?

Claim: DA algo converges in poly time

- Every man makes $\leq n$ proposals
- There could be at most n^2 proposals

  $O(n^2)$

Claim: DA algo always returns a perfect matching (i.e., all vertices are matched)

- No woman is matched to more than one man
- Every woman is either tentatively matched (OR) gets multiple prop. and keeps one.
- Once a woman is tent matched, never unmatched again.
- The algo terminates when each man is matched.

Claim: DA gives a pairwise stable matching

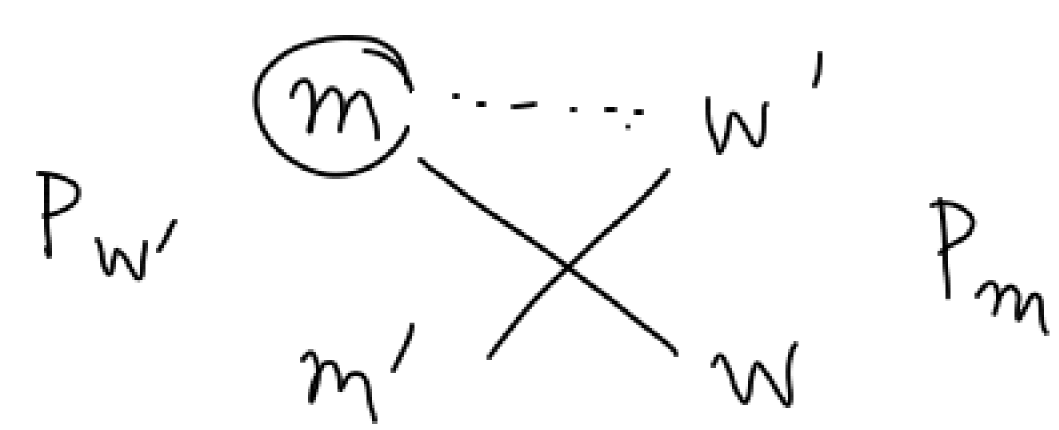
$\mu_P : M \to W$ $\mu_P(m) \leftarrow$ woman that m is matched to

$\mu_P(w) \leftarrow$ man ... w, ...

A matching is pairwise unstable if

$\exists$ P, and m, w, m', w' s.t.

$$\left.\begin{array}{l} \mu_P(m) = w \\ \mu_P(m') = w' \end{array}\right\} \begin{array}{l} \text{and} \\ w' \; P_m \; w \\ m \; P_{w'} \; m' \end{array}$$

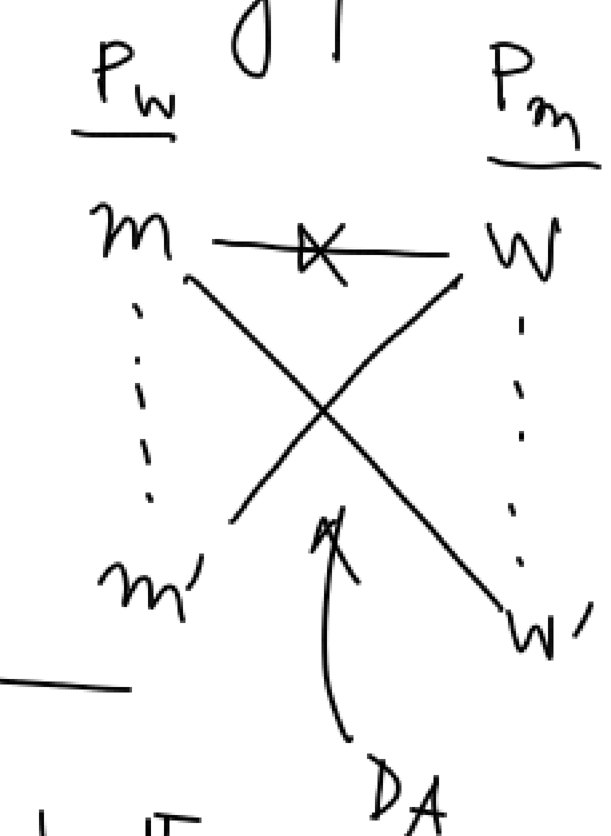


(m, w') is a blocking pair.

Proof: Suppose not,

$\exists$ P, blocking pair

(m, w)



Contradicts to the working principle of DA.