CS 217: Artificial Intelligence and Machine Learning	**Jan-Apr 2024**

Lecture 21: Normal Form Game Representation

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21.1 Is his loss, my win?

In the past few lectures we studied about zero sum games. But not all games have total utility as zero. That is to say, win for one may not mean loss for another. So let us delve into a more general set of games! However, we will begin by introducing some notations to aid us.

21.1.1 Notations

For a game, we will have certain number of players, and each player will have some strategy and some utility for a given set of moves. Following notations give us a compact way to express all of these.

- $\mathcal{N}$: set of players or agents.
  $\mathcal{N} = \{1, 2, \ldots, n\}$
- S_i : strategy set of player $i \in \mathcal{N}$
- s_i : startegy of a particular player i , $s_i \in S_i$
- $s = (s_1, s_2, \ldots s_n)$: strategy profile of n players
  $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$
  Thus, $s = (s_i, s_{-i})$
- $u_i : S_1 \times S_2 \cdots \times S_n \rightarrow \mathbb{R}$
  $u_i(s_i, s_{-i}) \in \mathbb{R}$ is the utility of player i

Now, let's understand these notations using the example of Arms race.

21.1.2 Example: Arms race

$1 \downarrow 2 \rightarrow$	Peace	War
Peace	$(5, 5)$	$(0, 10)$
War	$(10, 0)$	$(1, 1)$

Table 21.1: The payoff matrix for an arms race scenario.

In this example:

- $\mathcal{N} = \{1, 2\}$
- $S_1 = S_2 = \{Peace, War\}$
- $u_1(Peace, War) = 0$ and $u_2(Peace, War) = 10$

21.1.3 Normal form game representation

Normal form representation is a tuple of 3 elements used to represent a game.

$$\langle \mathcal{N}, (S_i)_{i \in \mathbb{N}}, (u_i)_{i \in \mathbb{N}} \rangle$$

21.2 Strategy, dominant or dominated?

21.2.1 Dominated strategy

Informally, dominated strategy is one which performs worse compared to some other strategy in every possible outcome. A weakly dominated strategy on the other hand might fare as good as the other strategy in some outcomes, but must be worse in at least one outcome. So we define it formally as follows:

21.2.1.1 Strictly Dominated

A strategy s_i' of player i is strictly dominated if there exists another strategy $s_i \in S_i$ such that for all $s_{-i} \in S_{-i}$, where $S_{-i} = \underset{i \neq j}{X} S_j$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$. In this case, $s_{i'}$ is strictly dominated by s_i.

21.2.1.2 Weakly Dominated

A strategy s_i' of player i is weakly dominated if there exists another strategy $s_i \in S_i$ such that for all $s_{-i} \in S_{-i}$, where $S_{-i} = \underset{i \neq j}{X} S_j$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and there exists $\tilde{s}_i \in S_{-i}$, $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$.

Let's take an example to understand this:

$1 \downarrow$ $2 \rightarrow$	D	E
A	$(5, 5)$	$(0, 5)$
B	$(5, 0)$	$(1, 1)$
C	$(4, 0)$	$(1, 1)$

Table 21.2: Strategic outcomes for Player 1 and Player 2.

$$\left.\begin{aligned} u_1(B,D) &= u_1(A,D) \\ u_1(B,E) &> u_1(A,E) \end{aligned}\right\} \quad \text{Thus, strategy } A \text{ is weakly dominated by strategy } B \text{ for player 1.} \tag{21.1}$$

$$\left.\begin{aligned} u_1(B,E) &= u_1(C,E) \\ u_1(B,D) &> u_1(C,D) \end{aligned}\right\} \quad \text{Thus, strategy } C \text{ is weakly dominated by strategy } B \text{ for player 1.} \tag{21.2}$$

$$\left.\begin{aligned} u_2(E,A) &= u_2(D,A) \\ u_2(E,B) &> u_2(D,B) \\ u_2(E,C) &> u_2(D,C) \end{aligned}\right\} \quad \text{Thus, strategy } D \text{ is weakly dominated by strategy } E \text{ for player 2.} \tag{21.3}$$

21.2.2 Dominant strategy

Following from the definitions we mentioned earlier, we define dominant strategy as a strategy that dominates every other strategy of the player. Domination can either be strict, or weak, and thus the strategy can either be strictly dominant or weakly dominant.

In the above example, B is a weakly dominant strategy for player 1.

21.3 Equilibrium

Intuitively, we can see that an equilibrium can be achieved when every player has some 'best' move irrespective of what the other player does. Thus, we state the following:

If both the players have strictly dominant (**SDS**) or weakly dominant strategies (**WDS**), the strategy profile of their SDS/WDS is called strictly/weakly dominant strategy equilibrium (**SDSE/WDSE**).

In the above example (B,E) is WDSE since both B and E are weakly dominant strategies. In the earlier example of arms race, (War, War) is SDSE.

Note:
For the equilibrium to be SDSE, all the player must have SDS. Existence of at least one player having a WDS but no SDS will lead to a WDSE and not SDSE.

21.3.1 Professor's Dilemma

We must now solve a dilemma. Professor is confused, whether he should put effort or not. Let us try to help the professor by guessing the utility that professor and student gains in each scenario.

$P \downarrow \; S \rightarrow$	**Listen**	**Sleep**
Effort	$(100, 100)$	$(-10, 0)$
No Effort	$(0, -10)$	$(0, 0)$

Table 21.3: Decision matrix for the Professor and Student scenario.

Sadly there are no dominant strategies for professor and students in the example of Professor's Dilemma.

However, there are two different type of equilibria as we discuss below.

$$(s_1^*, s_2^*) = (E, L), u_1(s_1^*, s_2^*) = 100 \tag{21.4}$$
$$(s_1', s_2^*) = (NE, L), u_1(s_1', s_2^*) = 0 \tag{21.5}$$
$$(s_1^*, s_2^*) = (E, L), u_2(s_1^*, s_2^*) = 100 \tag{21.6}$$
$$(s_1^*, s_2') = (E, S), u_2(s_1^*, s_2') = -10 \tag{21.7}$$

From the equations above it is clear that $u_1(s_1^*, s_2^*) > u_1(s_1', s_2^*)$ hence player 1 has no reason to unilaterally deviate from the action profile (s_1^*, s_2^*). Similarly, $u_2(s_1^*, s_2^*) > u_2(s_1^*, s_2')$ and therefore, player 2 also has no reason to unilaterally deviate from the action profile (s_1^*, s_2^*). The action profile (s_1^*, s_2^*) appears as a *self-enforced agreement* among the two players and is a stable point, which we call a *pure strategy Nash equilibrium.*

21.3.2 Pure Strategy Nash Equilibrium (Nash 1951)

Pure Strategy Nash Equilibrium **(PSNE)** is a strategy profile from which unilateral deviations (other players' actions are fixed and only one player moves) are not beneficial. A PSNE is a strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ such that,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i \quad \forall i \in \mathbb{N}$$

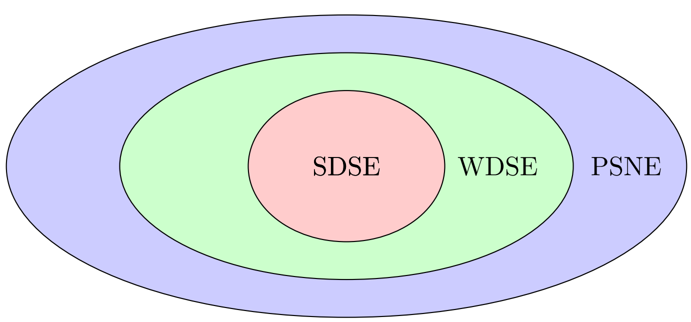
In the above example *(Effort, Listen)* is a PSNE because:

$$u_1(\textit{Effort, Listen}) > u_1(\textit{No Effort, Listen}) \quad \text{Unilateral deviation in strategy of player 1}$$
$$u_2(\textit{Effort, Listen}) > u_2(\textit{Effort, Sleep}) \quad \text{Unilateral deviation in strategy of player 2}$$

Homework: Do there exist other PSNEs in the above game? If so, find it.

21.3.3 Relation between WDSE, SDSE and PSNE

It is easy to verify that if (s_i^*, s_{-i}^*) is a WDSE, then (s_i^*, s_{-i}^*) is a PSNE, and it's clear that if (s_i^*, s_{-i}^*) is a SDSE, then (s_i^*, s_{-i}^*) is a WDSE.



That is to say SDSE $\implies$ WDSE $\implies$ PSNE, and the above figure shows the Venn diagram of games with such type of equilibrium, i.e.,

$$SDSE \subseteq WDSE \subseteq PSNE.$$

21.3.4 Transportation Problem

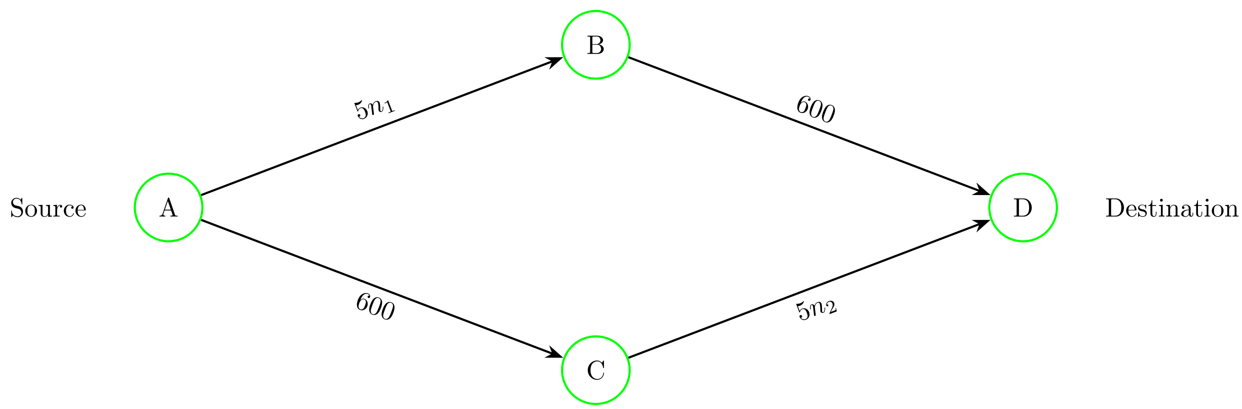


Figure 21.1: Transportation network

Consider the Figure 21.1. Here n_1 and n_2 represent the number of cycles passing through that path. The number labelling the arrows represent the time taken to traverse which we need to minimise. Let the total number of cycles be 100. Each person makes a decision, to traverse either the path ABD, or ACD, only after seeing how many people are already traversing a given path.

Observe that this is a Pure Strategy Nash Equilibrium (**PSNE**). For a person choosing ABD, ACD would never be any more beneficial. This is not a Weakly Dominant Strategy Equilibrium (WDSE), as it is not satisfied for all "strategy profiles." That is to say, any of the path ABD or ACD can be favourable depending on the choices of the previous students (players) In this equilibrium, the total time = 850 for 100 cycles.

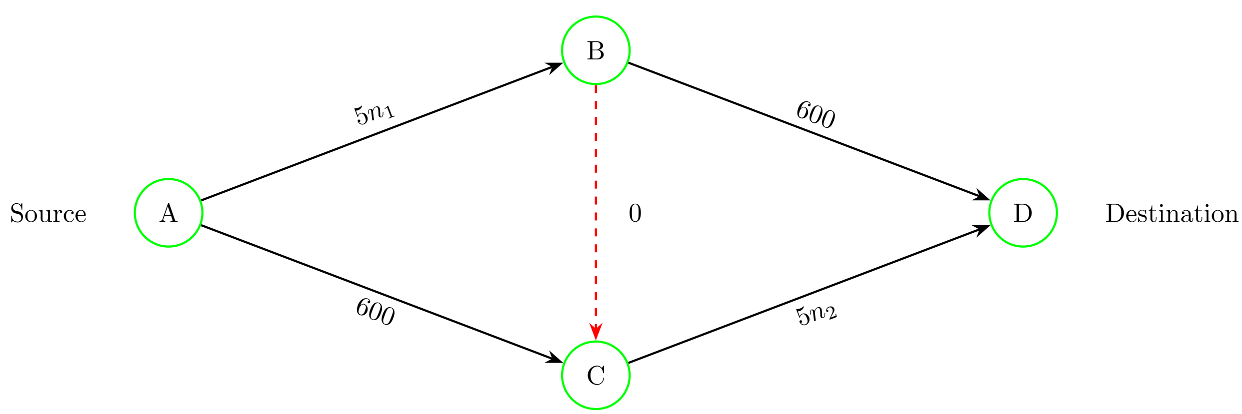


Figure 21.2: Transportation network with an added path

Now Consider a variation (Figure 21.2) where we introduce a new $B \to C$ path which can be reached in 0 time.

New equilibrium will be that everyone (all 100 cycles) choose $ABCD$.

This equilibrium is **SDSE** as irrespective of others strategies we adopt $ABCD$, and the inequalities are strict.

In this equilibrium, total time = 1000.

The interesting thing is, despite there being an additional choice, there has been an increase in total time taken! This is called the **Braess' Paradox** (as the time has increased here).

21.3.5 On the existence of PSNE

Recap (Two player zero sum games)

$$\overline{v} = \max_{s_2} \min_{s_1} u(s_1, s_2)$$

$$\underline{v} = \min_{s_2} \max_{s_1} u(s_1, s_2)$$

Lemma 21.1 $\overline{v} \geq \underline{v}$

Theorem 21.2 *A matrix game u has a PSNE (saddle point) if and only if* $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$*, where* s_1^* *and* s_2^* *are max-min and min-max strategies of players 1 and 2.*

$$s_1^* = \arg\max_{s_1} \min_{s_2} u(s_1, s_2)$$

$$s_2^* = \arg\min_{s_2} \max_{s_1} u(s_1, s_2)$$

See example in Lecture 20 and verify $\overline{v} = \underline{v}$.

Let us consider a simpler version of penalty shootout to demonstrate this.

$S \downarrow G \rightarrow$	Left	Right
Left	$(-1, 1)$	$(1, -1)$
Right	$(1, -1)$	$(-1, 1)$

Table 21.4: The payoff matrix for penalty shootout

It is easy to consider all the possible strategy profiles and verify that **PSNE** does not exist in the above case.

PSNE may not exist always!

21.4 Mixed Strategy

We can define a mixed action for a player using probabilities.
For example: $\sigma_2 = \left(\frac{4}{5}, \frac{1}{5}\right)$ player picks L with prob $\frac{4}{5}$ and R with prob $\frac{1}{5}$.

Thus, a mixed strategy is a probability distribution over the pure strategies.
Let us analyse the expected utilities, using some examples:

Consider

$$\sigma_2 = \left(\frac{4}{5}, \frac{1}{5}\right) \quad \sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$$

$$u_1(L, \sigma_2) = u_1(L, L) \times \frac{4}{5} + u_1(L, R) \times \frac{1}{5}$$

$$u_1(\sigma_1, \sigma_2) = u_1(L, L) \times \frac{2}{3} \times \frac{4}{5} + u_1(L, R) \times \frac{1}{3} \times \frac{1}{5} + u_1(R, L) \times \frac{1}{3} \times \frac{4}{5} + u_1(R, R) \times \frac{1}{3} \times \frac{1}{5}$$

In the example of penalty shootout above:

$$u_1(L, \left(\frac{4}{5}, \frac{1}{5}\right)) = -\frac{3}{5} \qquad u_1(R, \left(\frac{4}{5}, \frac{1}{5}\right)) = \frac{3}{5} \qquad \text{L is a better choice}$$

$$u_1(L, \left(\frac{1}{5}, \frac{4}{5}\right)) = \frac{3}{5} \qquad u_1(R, \left(\frac{1}{5}, \frac{4}{5}\right)) = -\frac{3}{5} \qquad \text{R is a better choice}$$

$$u_1(L, \left(\frac{1}{2}, \frac{1}{2}\right)) = 0 = u_1(R, \left(\frac{1}{2}, \frac{1}{2}\right))$$

$$u_1(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)) \geq u_1(\sigma_1, \left(\frac{1}{2}, \frac{1}{2}\right))$$

A strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a mixed strategy NE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i, \forall i \in N$$