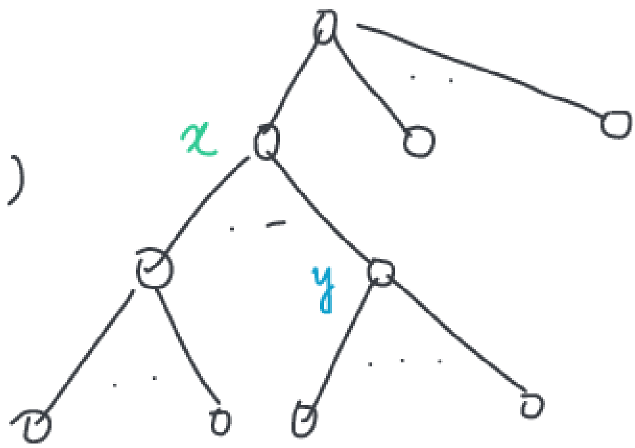
An early result of Game Theory (von Neumann, 1928)

In chess, one and only one of the following statements is true

1. W has a winning strategy
2. B has a winning strategy
3. Each player has a strategy guaranteeing a draw



Proof: Each vertex is a game situation

$\Gamma(x)$: subtree rooted at x (includes itself)

n_x : number of vertices in $\Gamma(x)$

y is a vertex in $\Gamma(x)$, $y \neq x$

$\Gamma(y)$ is a subtree of $\Gamma(x)$, $n_y < n_x$

$n_x = 1 \Rightarrow x$ is a terminal vertex

The proof is via induction on n_x

The theorem holds for $n_x = 1$, why?

if W king is removed, B wins

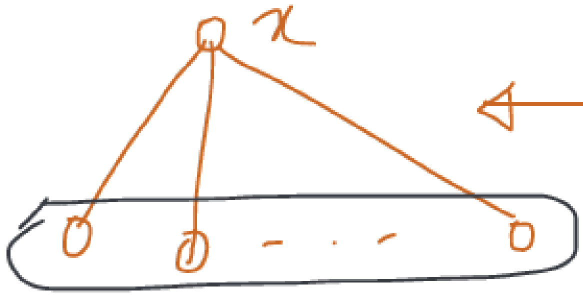
if B king is removed, W wins

if both kings present, but game ends — draw

Suppose x is a vertex with $n_x > 1$

Induction hypothesis: for all vertices $y \in \Gamma(y)$, s.t. $n_y < n_x$, the statement holds

in particular, $\boxed{\Gamma(y) \text{ is a subgame of } \Gamma(x)}$



WLOG
W moves

$C(x)$ = Vertices reachable from x in one step.

Case (i) if $\exists y_0 \in C(x)$, s.t. (1) is true in $\Gamma(y_0)$, Then (1) is true in $\Gamma(x)$
W just picks that

Case (ii) if $\forall y \in C(x)$, (2) is true, Then (2) is true in $\Gamma(x)$
B sees that action and picks The appropriate action to win.

Case (iii)

- (i) does not hold, W does not have a winning strategy in any $y \in C(x)$
  since induction hypothesis holds for every $y \in C(x)$, either B has a winning strategy or both have draw-guaranteeing strategy
- (ii) doesn't hold, $\exists y' \in C(x)$ where B doesn't have a winning strategy
  since (i) doesn't hold either, W can't guarantee a win in y'
  - hence they both have strategies guaranteeing a draw.
    W picks The action to reach y'.
    B picks action that guarantees a draw or win.

This concludes The proof.

Exercise: prove this when The length of game is infinite, (ex 1.3 MSZ)