Earlier matrix game examples

u	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
minmax	1	1	

$\bar{v} = 1 > -1 = \underline{v}$

PSNE doesn't exist

u	L	C	R	maxmin
T	3	-5	2	-5
M	1	4	1	1
B	6	-3	-5	-5
minmax	6	4	1	

$\bar{v} = 1 = \underline{v}$

PSNE exists

Define $s_1^* \in \arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$: maxmin strategy of 1

$s_2^* \in \arg \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$: minmax strategy of 2

Theorem: A matrix game u has a PSNE (saddle point) if and only if $\bar{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively. In particular, (s_1^*, s_2^*) is a PSNE.

Proof: $(\Rightarrow)$ i.e., PSNE $\Rightarrow \bar{v} = \underline{v} = u(s_1^*, s_2^*)$

Say the PSNE is (s_1^*, s_2^*), i.e., $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$

$$\Rightarrow u(s_1^*, s_2^*) \geqslant \max_{t_1 \in S_1} u(t_1, s_2^*)$$

$$\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2)$$, since s_2^* is a specific strategy

$$= \bar{v}$$

Similarly, using the same argument for player 2, we get

$\underline{v} \geq u(s_1^*, s_2^*)$, for player 2 utility $u_2 = -u$

But $\overline{v} \geq \underline{v}$ [from previous lemma]

Hence, $u(s_1^*, s_2^*) \geq \overline{v} \geq \underline{v} \geq u(s_1^*, s_2^*)$

$\Rightarrow u(s_1^*, s_2^*) = \overline{v} = \underline{v}$, also implies that the maxmin for 1 and minmax for 2 are s_1^* and s_2^* resp.

$(\Leftarrow)$ given $u(s_1^*, s_2^*) = \overline{v} = \underline{v}$ $= v$ (say), s_1^*, s_2^* are maxmin and minmax resp. for 1 and 2.

$$u(s_1^*, s_2) \geq \min_{t_2 \in S_2} u(s_1^*, t_2) \quad : \text{by defn of min}$$

$\forall s_2 \in S_2$

$$= \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \quad : \text{since } s_1^* \text{ is the maxmin strategy for 1.}$$

$$= v \text{ (given)}$$

Similarly show, $u(s_1, s_2^*) \leq v$ $\quad \forall s_1 \in S_1$

but $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE