MSNE characterization theorem

<u>Theorem</u>: A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff $\forall i \in N$

① $u_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in \delta(\sigma_i^*)$

② $u_i(s_i, \sigma_{-i}^*) \geqslant u_i(s_i', \sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$

Observations:

① $$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

maximizing w.r.t. a distribution $\equiv$ whole probability mass at max

② $$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$

$(\sigma_i^*, \sigma_{-i}^*)$ MSNE

the maximizer must lie in $\delta(\sigma_i^*)$ – if no maximizer in $\delta(\sigma_i^*)$ then put all probability mass on that $s_i' \notin \delta(\sigma_i^*)$ that has the maximum value of the utility – $(\sigma_i^*, \sigma_{-i}^*)$ is not a MSNE.

<u>Proof</u>: $(\Rightarrow)$ given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \quad \text{- - - -} \quad ① \end{aligned}$$

by definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i)\ u_i(s_i, \sigma_{-i}^*)$$

$$= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\ u_i(s_i, \sigma_{-i}^*) \quad \text{-----} \quad (2)$$

(1) and (2) are equal - max is equal to positive weighted average

- can happen only when all values are same. proves condition 1

for condition 2: suppose for contradiction

$$\exists\ s_i \in \delta(\sigma_i^*) \text{ and } s_i' \notin \delta(\sigma_i^*)$$

$$\text{s.t. } u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$$

shift the probability mass $\sigma_i^*(s_i)$ to s_i', this new mixed strategy gives a strict better utility - contradiction to MSNE.

($\Leftarrow$) Given the two conditions of the theorem hold

let $u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*)$, $\forall\ s_i \in \delta(\sigma_i^*)$ - condition 1

note $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$ - condition 2

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\ u_i(s_i, \sigma_{-i}^*) \quad \text{- by defn. of } \delta(\sigma_i^*)$$

$$= m_i(\sigma_{-i}^*) \quad \text{- previous conclusion}$$

$$= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \quad \text{- previous conclusion}$$

from the observation

$$= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*),\ \forall \sigma_i \in \Delta(S_i)$$

algorithmic way to find MSNE