MSNE characterization theorem to algorithm

NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

All possible supports of $S_1 \times S_2 \times \cdots \times S_n$

number $= K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$

for every support profile $X_1 \times X_2 \cdots \times X_n$, where $X_i \subseteq S_i$

solve the following feasibility program

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \quad \forall s_i \in X_i, \forall i \in N \quad \text{— cond 1}$$

$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N \quad \text{— cond 2}$$

$$\sigma_j(s_j) \geq 0, \ \forall s_j \in S_j, \forall j \in N, \text{ and } \sum_{s_j \in S_j} \sigma_j(s_j) = 1, \ \forall j \in N.$$

feasibility program with variables w_i, $i \in N$, $\sigma_j(s_j)$, $s_j \in S_j$, $j \in N$.

Remarks : this is not a linear program unless $n = 2$

For general games, there is no poly-time algorithm

Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs]

Daskalakis, Goldberg, Papadimitriou "The complexity of computing a Nash equilibrium" 2009.

MSNE and dominance

The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

	L	R
T	4, 1	2, 5
M	1, 3	6, 2
B	2, 2	3, 3

Dominated strategy in this game?

domination can also be via mixed strategy

weak dominated strategy removal can remove equilibrium

for strictly dominated strategies

Theorem: If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$, then in every MSNE of the game, s_i is chosen with probability zero.

So, can remove without loss of equilibrium.

Existence of MSNE

Finite game: number of players and the strategies are finite

Theorem (Nash 1951)

Every finite game has a (mixed) Nash equilibrium.

Proof requires a few tools and a result from real analysis

- A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda) x \in S$
- A set $S \subseteq \mathbb{R}^n$ is closed if it contains all its limit point (points whose every neighborhood contains a point in S — a set not closed $[0,1)$ — every ball of radius $\epsilon > 0$ around 1 has a member of $[0,1)$, but 1 is not in the set $[0,1)$)
- A set $S \subseteq \mathbb{R}^n$ is bounded if $\exists x_0 \in \mathbb{R}^n$ and $R \in (0,\infty)$ s.t.

  $\forall x \in S, \; \|x - x_0\|_2 < R$
- A set $S \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

A result from real analysis (without proof)

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is convex and compact and $T : S \to S$, is continuous

Then T has a fixed point, i.e., $\exists x^* \in S$ s.t. $T(x^*) = x^*$.