Mixed strategy equivalent of behavioral strategy

Theorem (6.11 of MSZ)
Consider an IIEFG s.t. every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy iff each information set of a player intersects every path emanating from the root at most once.

Behavioral strategy equivalent of mixed strategy

To formalize (i.e., set the conditions when the equivalence holds), we need to formalize the forgetfulness of the player

- saw few examples of players' forgetfulness
- our conditions need to ensure that none of those forgetfulness happens

Definition (Choice of same action at an information set)

Let $X = (x^0, x^1, \dots, x^K)$ and $\hat{X} = (x^0, \hat{x}^1, \dots, \hat{x}^L)$ be two paths in the game tree
Let I_i^j be an information set of player i that intersects these two paths only at one vertex, say x^k and x^l respectively.
These two paths choose the same action at information set I_i^j if

- $k < K$ and $l < L$
- actions x^k leading to x^{k+1} and $\hat{x}^l$ leading to $\hat{x}^{l+1}$ are identical denoted by $a_i(x^k \to x^{k+1}) = a_i(\hat{x}^l \to \hat{x}^{l+1})$

"leading to" may not be a relation between parent and child nodes it can be any descendant of the former since the path is unique in a tree.

Games with Perfect Recall

Definition

Player i has perfect recall if the following conditions are satisfied

1. Every information set of player i intersects every path from the root to a leaf at most once.

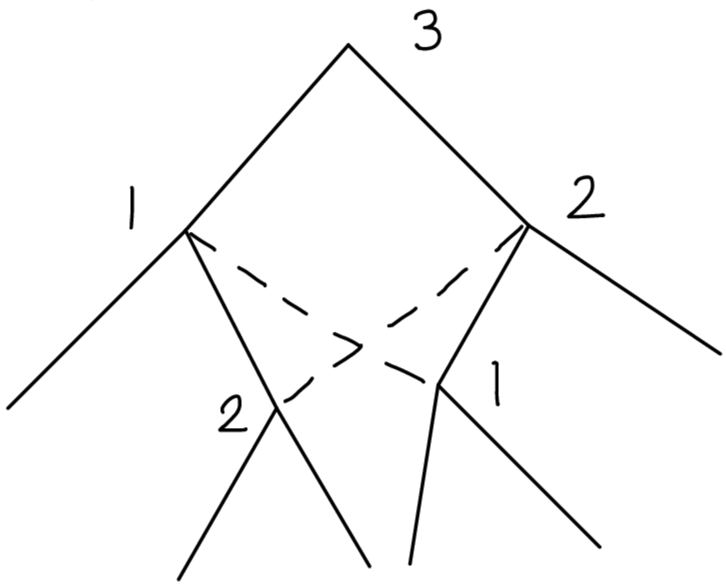
2. Every two paths that end in the same information set of player i pass through the same information sets of i in the same order and in every such information set the two paths choose the same action.

Rephrasing: for every I_i^j of i and every pair of vertices x and $x' \in I_i^j$ if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$, and $x_i'^1, x_i'^2, \ldots, x_i'^{L'} = x'$ respectively for the two paths from root to x and x'
then
1. $L = L'$
2. $x_i^\ell, x_i'^\ell \in I_i^k$ for some k, and
3. $a_i(x_i^\ell \to x_i^{\ell+1}) = a_i(x_i'^\ell \to x_i'^{\ell+1})$, $\forall \ell = 1, 2, \ldots, L-1$.

A game is of perfect recall if every player has perfect recall.

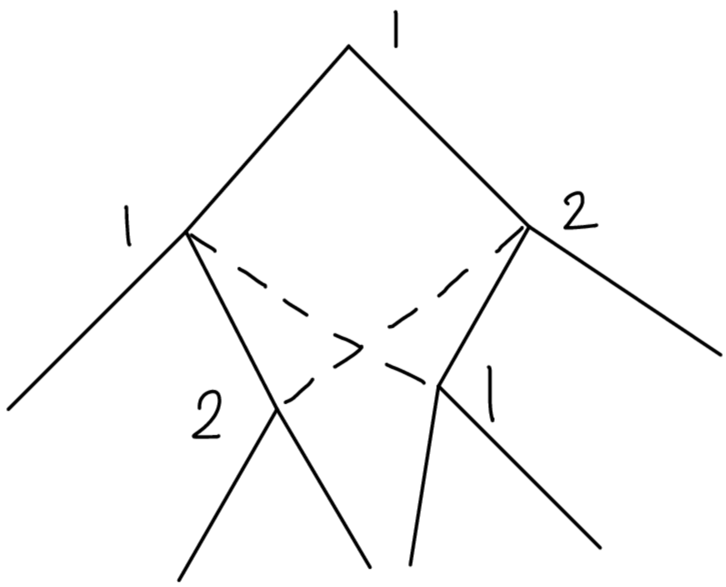
Note: definition of perfect recall subsumes the condition the theorem where every behavioral strategy has equivalent mixed strategy (point 1)

Examples



This example satisfies the conditions of the definition

game with perfect recall



Player 1 takes two different actions at the first information set to reach two different vertices of the second information set

game with imperfect recall

Implications of perfect recall

Let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses actions leading to x — i.e., intersections of members of S_i with the path from root to x.

Theorem: If i is a player with perfect recall and x and x' are two vertices in the same information set of i. Then $S_i^*(x) = S_i^*(x')$.

The above conclusion comes from the same sequence of information sets and same actions. The next implication gives the equivalence of mixed and behavioral strategies.

Theorem (Kuhn 1957)

In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of i, there exists a behavioral strategy

The converse is already true (beh has equiv mixed) since the sufficient condition for that is already subsumed in the definition of perfect recall.

Proof: reading exercise (MSZ Theorem 6.15)

Remarks: the proof is constructive. It starts with a mixed strategy and constructs the behavioral strategies s.t. the probabilities of reaching a leaf are same. The arguments show that such a construction is always possible because of perfect recall.