Equilibrium concepts in Bayesian games

Ex-ante: before observing own type

Nash equilibrium (σ^*, P): $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$, $\forall \sigma_i'$, $\forall i \in N$

Ex-interim: after observing own type

Bayesian equilibrium (σ^*, P)

$$u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^* \mid \theta_i), \quad \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i$$

The RHS of the definition can be replaced by a pure strategy a_i, $\forall a_i \in A_i$
The reason is exactly same as that of MSNE (these definitions are equivalent)

NE notion takes expectation over $P(\theta)$, BE notion takes expectation over $P(\theta_{-i} \mid \theta_i)$

Equivalence of the two equilibrium concepts

Theorem: In finite Bayesian games, a strategy profile is a Bayesian equilibrium iff it is a Nash equilibrium.

Proof: $(\Rightarrow)$ Suppose (σ^*, P) is a BE, consider

$$u_i(\sigma_i', \sigma_{-i}^*) = \sum_{\theta_i \in \Theta_i} P(\theta_i)\, u_i(\sigma_i'(\theta_i), \sigma_{-i}^* \mid \theta_i)$$

$$\overset{\text{BE}}{\leq} \sum_{\theta_i \in \Theta_i} P(\theta_i)\, u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i) = u_i(\sigma_i^*, \sigma_{-i}^*).$$

$(\Leftarrow)$ Proof by contradiction. Suppose (σ^*, P) is not a BE, i.e., there exists some $i \in N$, some $\theta_i \in \Theta_i$, and some $a_i \in A_i$, s.t.

$$u_i(a_i, \sigma_{-i}^* \mid \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i)$$

Construct the strategy $\hat{\sigma}_i$, $\hat{\sigma}_i(\theta_i') = \sigma_i^*(\theta_i')$ $\forall \theta_i' \in \Theta_i \setminus \{\theta_i\}$

$\hat{\sigma}_i(\theta_i)[a_i] = 1$, $\hat{\sigma}_i(\theta_i)[b_i] = 0 \ \forall\, b_i \in A_i \setminus \{a_i\}$

Then, $u_i(\hat{\sigma}_i, \sigma^*_{-i}) = \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i)$

$$= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i) + P(\theta_i)\, \underbrace{u_i(\hat{\sigma}_i(\theta_i), \sigma^*_{-i} \mid \theta_i)}_{> u_i(\sigma^*_i(\theta_i), \sigma^*_{-i} \mid \theta_i)}$$

$$> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i) + P(\theta_i)\, u_i(\sigma^*_i(\theta_i), \sigma^*_{-i} \mid \theta_i)$$

$$= u_i(\sigma^*_i, \sigma^*_{-i})$$

Hence $(\sigma^*_i, \sigma^*_{-i})$ is not a Nash equilibrium.

Existence of Bayesian equilibrium

Theorem: Every finite Bayesian game has a Bayesian equilibrium

[finite Bayesian game: set of players, action set, type set are finite]

proof idea: transform the Bayesian game into a complete information game treating each type a player, and invoke Nash Theorem for existence of equilibrium - which is a BE in the original game. [see addendum for details]