Bayesian equilibria in Bayesian games

Sealed bid auction

Two players, both willing to buy an object. Their values and bids lie in $[0,1]$

allocation function: $O_1(b_1,b_2) = I\{b_1 \geq b_2\}$; $O_2(b_1,b_2) = I\{b_2 > b_1\}$

beliefs: $f(\theta_2|\theta_1)=1, \ \forall \theta_1, \theta_2$ $\qquad f(\theta_1,\theta_2)=1, \ \forall(\theta_1,\theta_2) \in [0,1]^2$

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① First price auction: if $b_1 \geq b_2$, player 1 wins and pays her bid
ow, player 2 wins and pays her bid

$$u_1(b_1,b_2,\theta_1,\theta_2) = (\theta_1 - b_1) I\{b_1 \geq b_2\}$$

$$u_2(b_1,b_2,\theta_1,\theta_2) = (\theta_2 - b_2) I\{b_1 < b_2\}$$

$b_1 = s_1(\theta_1)$, $b_2 = s_2(\theta_2)$, assume $s_i(\theta_i) = \alpha_i \theta_i$, $\alpha_i > 0$, $i=1,2$.

To find the BE, we need to find the s_i^* (or α_i^*) that maximizes the ex-interim utility of player i

$$\max_{\sigma_i} \ u_i(\sigma_i, \underline{\sigma}_i^* | \theta_i),$$

For player 1, this reduces to:

$$\max_{b_1 \in [0,\alpha_2]} \int_0^1 f(\theta_2|\theta_1)(\theta_1 - b_1) I\{b_1 \geq \alpha_2\theta_2\} \, d\theta_2$$

(since $\theta_2 \in [0,1]$ b_1 never needs to be larger than α_2)

$$= \max_{b_1 \in [0,\alpha_2]} (\theta_1 - b_1)\frac{b_1}{\alpha_2} \Rightarrow b_1^* = \begin{cases} \frac{\theta_1}{2} & \text{if } \alpha_2 > \frac{\theta_1}{2} \\ \alpha_2 & \text{ow} \end{cases}$$

$$s_1^*(\theta_1) = \min\left\{\frac{\theta_1}{2}, \alpha_2\right\}, \quad s_2^*(\theta_2) = \min\left\{\frac{\theta_2}{2}, \alpha_1\right\}$$

if $\alpha_1 = \alpha_2 = 1/2$, then $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ is a BE

In the Bayesian game induced by uniform prior on first price auction, bidding half the true value is a Bayesian equilibrium.

② Second price auction: highest bidder wins but pays the second highest bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)\, I\{b_1 \geqslant b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)\, I\{b_1 < b_2\}$$

Player 1's bidding problem is to maximize

$$\int_0^1 f(\theta_2|\theta_1)\,(\theta_1 - s_2(\theta_2))\, I(b_1 \geqslant s_2(\theta_2))\, d\theta_2$$

$$= \int_0^1 1 \cdot (\theta_1 - \alpha_2\theta_2)\, I\left(\theta_2 \leqslant \frac{b_1}{\alpha_2}\right) d\theta_2$$

$$= \frac{1}{\alpha_2}\left(b_1\theta_1 - \frac{\theta_1^2}{2}\right) \Rightarrow \text{maximized when } b_1 = \theta_1$$

similarly for $b_2 = \theta_2$.

If the distributions of θ_1 and θ_2 were arbitrary but independent the maximization problem would have been

$$\int_0^{b_1/\alpha_2} f(\theta_2)\,(\theta_1 - \alpha_2\theta_2)\, d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{b_1/\alpha_2} \theta_2 f(\theta_2)\, d\theta_2$$

differentiating wrt b_1, we get

$$\theta_1 \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \frac{b_1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) \cdot \frac{1}{\alpha_2} = 0$$

$$\Rightarrow \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right)(b_1 - \theta_1) = 0 \Rightarrow b_1 = \theta_1 \text{ (similar for 2), if } f\left(\frac{b_1}{\alpha_2}\right) > 0$$

For any independent, positive prior, bidding true type is a BE of the induced Bayesian game in second price auction.