Theorem: An SCF f is strategyproof (SP) iff it is monotone (MONO).

Note: The proof technique, will be used later as well.

Proof: SP $\Rightarrow$ MONO, consider the "if" condition of MONO

P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i')$ $\forall i \in N$

Break the transition from P to P' into n stages

$$\underset{P = P^{(0)}}{(P_1\, P_2 \cdots P_n)} \to \underset{P^{(1)}}{(P_1'\, P_2 \cdots P_n)} \to \underset{P^{(2)}}{(P_1'\, P_2' \cdots P_n)} \to \underset{P^{(k)}}{(P_1' \cdots P_k'\, P_{k+1} \cdots P_n)} \to \cdots \underset{P^{(n)} = P'}{(P_1' \cdots P_n')}$$

Claim: $f(P^{(k)}) = a$, $\forall k = 1, \ldots, n$

Suppose not, i.e., $\exists\, P^{(k-1)}, P^{(k)}$ s.t. $f(P^{(k-1)}) = a$, $f(P^{(k)}) = b \neq a$

$P_1' \cdots P_{k-1}'\, P_k \cdots P_n$ $\qquad$ $P_1' \cdots P_{k-1}'\, P_k' \cdots P_n$

a ← position has weakly bettered

outcome a $\qquad$ outcome b

there can be three cases:

$a\, P_k\, b$ and $a\, P_k'\, b$ → voter k misreports $P_k' \to P_k$

$b\, P_k\, a$ and $b\, P_k'\, a$ → voter k misreports $P_k \to P_k'$

$b\, P_k\, a$ and $a\, P_k'\, b$ → voter k misreports in both

contradiction to f SP.

SP $\Leftarrow$ MONO, we will prove !SP $\Rightarrow$!MONO

suppose not, i.e., f is !SP but MONO

!SP implies that $\exists\, i, P_i, P_i', P_{-i}$ s.t. $\underbrace{f(P_i', P_{-i})}_{=:\, b}\ P_i\ \underbrace{f(P_i, P_{-i})}_{=:\, a}$

hence $b \, P_i \, a$. construct P'' s.t. $P''_{-i} = P_{-i}$.

$P_i''(1) = b$, $P_i''(2) = a$

P_i''	P_{-i}
b	$\vdots$
a	
$\vdots$	

Consider two transitions

① $(P_i, P_{-i}) \to (P_i'', P_{-i})$

$$D(a, P_i) \subseteq D(a, P_i'') \underset{\text{MONO}}{\Rightarrow} f(P_i'', P_{-i}) = a$$

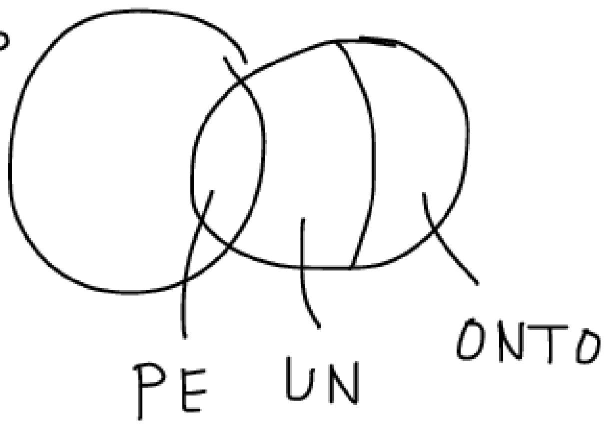
② $(P_i', P_{-i}) \to (P_i'', P_{-i})$

$$D(b, P_i') \subseteq D(b, P_i'') \underset{\text{MONO}}{\Rightarrow} f(P_i'', P_{-i}) = b \quad \text{(contradiction)}$$

This concludes the proof.

Lemma: If an SCF f is MONO and ONTO, then f is PE.

MONO $\Leftrightarrow$ SP



Proof: Suppose not, i.e., f is MONO and ONTO but not PE

then $\exists \, a, b, P$ s.t., $b \, P_i \, a \;\; \forall i \in N$ but $f(P) = a$.

ONTO: $\exists \, P'$ s.t. $f(P') = b$.

Construct P'' s.t. $P_i''(1) = b$, $P_i''(2) = a$, $\forall \, i \in N$

P''			
b	b	$\cdots$	b
a	a		a
$\vdots$	$\vdots$		$\vdots$

Clearly $D(b, P_i') \subseteq D(b, P_i'') \;\; \forall i \in N$

$$\underset{\text{MONO}}{\Rightarrow} f(P'') = b$$

Also $D(a, P_i) \subseteq D(a, P_i'') \ \forall i \in N$

$\underset{\text{MONO}}{\Rightarrow} \quad f(P'') = a$ (contradiction). Hence proved.

Corollary: f is SP + PE $\Leftrightarrow$ f is SP + UN $\Leftrightarrow$ f is SP + ONTO

Gibbard-Satterthwaite Theorem (G73, S75)

Suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial.

The statements with f is PE/UN and SP are equivalent.