Median voter SCF:

An SCF $f: \mathcal{S}^n \to A$ is a median voter SCF if there exists $B = \{y_1, y_2, \ldots, y_{n-1}\}$ s.t. $f(P) = \text{median}(B, \text{peaks}(P))$ for all preference profiles $P \in \mathcal{S}$.

[median wrt $<$]

The points in B are called the peaks of "phantom voters".

Note: B is fixed for f and does not change with P.

Why phantom voters?

$f^{\text{leftmost}} \equiv (B_{\text{Left}}, \text{peaks}(P))$; $B_{\text{Left}} = \{y_L, \ldots, y_L\}$

if all phantom peaks are on the left, it corresponds to leftmost peak SCF. Similarly, $f^{\text{rightmost}}(\cdot)$ can be found in a similar way. phantom voters give a complete description of the SCFs.

Theorem (Moulin 1980): Every median voter SCF is strategyproof.

Proof sketch: argue that if $f(P) = a$ and a player has a peak $P_i(1)$ to the left of a, it has no benefit by misreporting the peak to be on the right of a, which is the only way of changing the outcome of f. Similar for $P_i(1)$ on the right of a.

Note: mean does not have this property.

Claim: Let $p_{\min}$ and $p_{\max}$ are the leftmost and rightmost peaks of P according to $<$, then f is PE iff $f(P) \in [p_{\min}, p_{\max}]$

Proof: $\Rightarrow$ Suppose $f(P) \notin [p_{min}, p_{max}]$. WLOG, $f(P) < p_{min}$. Then every agent prefers p_{min} over $f(P)$, i.e., $f(P)$ is dominated. Hence $f(P)$ is not PE.

$\Leftarrow$ If $f(P) \in [p_{min}, p_{max}]$, then the condition $b \, P_i \, f(P) \; \forall i \in N$ never occurs. In other words, there does not exist an alternative b that Pareto dominates $f(P)$. Hence $f(P)$ is PE.

Consider monotonicity (MONO). The results similar to unrestricted preferences hold here too, but the proofs differ since we cannot construct preferences as freely as before.

Thm: f is SP $\Rightarrow$ f is MONO.

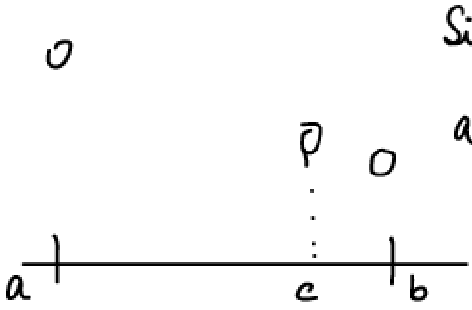
This proof is similar to the previous one. To prove the reverse implication one needs to argue why the construction is valid in the single peaked domain. (Or provide counterexample)

Thm: Let $f: \mathcal{S}^n \to A$ is a SP SCF. Then,

$$f \text{ is ONTO} \Leftrightarrow f \text{ is UN} \Leftrightarrow f \text{ is PE}$$

Proof: We know PE $\Rightarrow$ UN $\Rightarrow$ ONTO. To prove the above implication, we need to show that ONTO $\Rightarrow$ PE when f is SP.

Suppose, for contradiction, f is SP and ONTO, but not PE. Then $\exists \, a, b \in A$ s.t. $a \, P_i \, b \; \forall i \in N$ but $f(P) = b$.



Since preferences are single peaked, $\exists$ another alternative $c \in A$, which is a neighbor of b s.t. $c \, P_i \, b \; \forall i \in N$. c can be a itself.

ONTO $\Rightarrow \exists P'$ s.t. $f(P') = c$.

Construct P'' s.t. $P_i''(1) = c$, $P_i''(2) = b$, $\forall i \in N$.

$P \to P''$, MONO $\Rightarrow f(P'') = b$, $P' \to P''$ MONO $\Rightarrow f(P'') = c$.

We are interested in non-dictatorial SCFs.

Anonymity: (outcome insensitive to agent identities)

Permutation of agents $\sigma : N \to N$.

We apply a permutation σ to a profile P to construct another profile as: The preference of i goes to agent $\sigma(i)$ in the new profile. Denote this new profile as P^σ.

Example: $N = \{1, 2, 3\}$, σ: $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$

P_1	P_2	P_3
a	b	b
b	a	c
c	c	a

P_1^σ	P_2^σ	P_3^σ
b	a	b
c	b	a
a	c	c

The social outcome should not alter due to agent renaming.

Defn: An SCF $f : \mathcal{S}^n \to A$ is anonymous (ANON) if for every profile P and for every permutation of the agents σ,

$$f(P^\sigma) = f(P).$$

Example of a non-anonymous SCF?