Task allocation domain (related but different than single-peaked)

Unit amount of task to be shared among n agents

Agent i gets a share $s_i \in [0,1]$ of the job, $\sum_{i \in N} s_i = 1$.

Agent payoff: every agent has a most preferred share of work.

Example: The task has rewards – wages per unit time $= w$

if agent i works for t_i time then gets $w t_i$

the task also has costs, e.g., physical tiredness / less free time etc.

let the cost is quadratic $= c_i t_i^2$

net payoff $= w t_i - k_i t_i^2 \Rightarrow$ maximized at $t_i^* = \frac{w}{2c_i}$

and monotone decreasing on both sides.

This is single-peaked over the share of the task and not over the alternatives. Suppose, two alternatives are $(0.2, 0.4, 0.4)$ and $(0.2, 0.6, 0.2)$ – player 1 likes both of them equally.

There can't be a single common order over the alternatives s.t. the preferences are single-peaked for all.

Denote this domain of task allocation with T (single peaked over task share)

SCF: $f : T^n \to A$,

Let $P \in T^n$, $f(P) = (f_1(P), f_2(P), \cdots, f_n(P))$

$$f_i(P) \in [0,1], \forall i \in N; \quad \sum_{i \in N} f_i(P) = 1$$

Player i has a peak p_i over the share of task.

Pareto Efficiency: An SCF f is PE if there does not exist another share of task that is weakly preferred by all agents and strictly preferred by at least one, i.e.,

$$\nexists a \in A \text{ s.t. } a R_i f(P), \forall i \in N \text{ and } \exists j \text{ s.t. } a P_j f(P)$$

Implications:

① $\sum_{i \in N} p_i = 1$, allocate tasks according to the peaks of the agents. This is the unique PE.

② $\sum_{i \in N} p_i > 1$, $\exists k \in N$ s.t. $f_k(P) < p_k$.

Q: Can there be an agent j s.t. $f_j(P) > p_j$ if f is PE?

If so, increasing k's share of task and reducing j's makes both players strictly better off. Therefore

$$\forall j \in N, \quad f_j(P) \leq p_j.$$

③ If $\sum_{i \in N} p_i < 1$, similarly $\forall j \in N, f_j(P) \geq p_j$.

Anonymity: (if agent preferences are permuted, the shares will also get permuted accordingly.

$$f_{\sigma(j)}(P^{\sigma}) = f_j(P)$$

$N = \{1,2,3\}$, $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$

$P = (0.7, 0.4, 0.3) \Rightarrow P^{\sigma} = (0.3, 0.7, 0.4)$

$f_1(0.7, 0.4, 0.3) = f_2(0.3, 0.7, 0.4)$

Candidate SCFs:

Serial dictatorship: A predetermined sequence of the agents is fixed. Each agent is given either his peak share or a leftover share. If $\sum p_i < 1$, then the last agent is given the leftover share.

Properties: PE, SP, but not ANON. Also quite unfair for the last agent.

Proportional: Every player is assigned a share that is c times their peaks, s.t. $c \sum_{i \in N} p_i = 1$

overload if $\sum p_i < 1$, underload if $\sum p_i > 1$.

Q: Is it ANON, PE, SP?

Suppose peaks are 0.2, 0.3, 0.1 for 3 players, $c = \frac{1}{0.6}$

player 1 gets $\frac{1}{3}$ (more than 0.2)

if the report is 0.1, 0.3, 0.1, $c = \frac{1}{0.5}$, player 1 gets 0.2.