Pareto optimality in Quasi-linear domain

Defn: A mechanism $(f, (p_1, \dots, p_n))$ is Pareto Optimal if at every type profile $\theta \in \Theta$, there does not exist an allocation $b \neq f(\theta)$ and payments $(\pi_1, \dots, \pi_n)$ with $\sum_{i \in N} \pi_i \geq \sum_{i \in N} p_i(\theta)$ s.t.

$$v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - p_i(\theta), \quad \forall i \in N.$$

with the inequality being strict for some $i \in N$.

Pareto optimality is meaningless if there is no restriction on the payment. One can always put excessive subsidy to every agent to make everyone better off. So, the condition requires to spend at least the same budget.

Theorem: A mechanism $(f, (p_1, \dots, p_n))$ is Pareto optimal iff it is allocatively efficient.

Proof: $(\Rightarrow)$ we'll prove $!AE \Rightarrow !PO$

$$!AE \Rightarrow \exists\, b \neq f(\theta) \text{ s.t. } \sum_{i \in N} v_i(b, \theta_i) > \sum_{i \in N} v_i(f(\theta), \theta_i)$$

for some θ

$$\text{let } \delta = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > 0.$$

Consider payment $\pi_i = v_i(b, \theta_i) - v_i(f(\theta), \theta_i) + p_i(\theta) - \delta/n$

$$\text{hence, } [v_i(b, \theta_i) - \pi_i] - [v_i(f(\theta), \theta_i) - p_i(\theta)] = \delta/n > 0 \quad \forall i \in N$$

also $\sum_{i \in N} \pi_i = \sum_{i \in N} p_i(\theta)$. Hence f is not PO.

$\Leftarrow$ again we prove $!PO \Rightarrow !AE$

$!PO$, $\exists\, b, \underline{\pi}$ s.t. $\sum_{i\in N} \pi_i \geqslant \sum_{i\in N} p_i(\theta)$

$$v_i(b,\theta_i) - \pi_i \geqslant v_i(f(\theta),\theta_i) - p_i(\theta) \quad \forall i \in N$$

strict for some $j \in N$.

summing over the second inequality,

$$\sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} \pi_i > \sum_{i\in N} v_i(f(\theta),\theta_i) - \sum_{i\in N} p_i(\theta)$$

$$\Rightarrow \sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i) > \sum_{i\in N} \pi_i - \sum_{i\in N} p_i(\theta) \geqslant 0$$

$\Rightarrow$ f is $!AE$.

Allocative efficient rule is implementable

$$f^{\text{eff}}(\theta) \in \underset{a\in A}{\text{argmax}} \sum_{i\in N} v_i(a,\theta_i)$$

Consider the following payment:

$$p_i^G(\theta_i, \underline{\theta}_{-i}) = h_i(\underline{\theta}_{-i}) - \sum_{j\neq i} v_j(f^{\text{eff}}(\theta_i, \underline{\theta}_{-i}), \theta_j).$$

[Groves payment]

where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function.

Example: Single indivisible item allocation. $N = \{1,2,3,4\}$

$\theta_1 = 10$, $\theta_2 = 8$, $\theta_3 = 6$, $\theta_4 = 4$, when they get the object, zero

Otherwise. Let $h_i(\underline{\theta}_{-i}) = \min \underline{\theta}_{-i}$

if everyone reports their true type, The values of h_i are

$h_1 = 4, \; h_2 = 4, \; h_3 = 4, \; h_4 = 6$

The efficient allocation gives the item to agent 1.

$p_1 = 4 - 0 = 4, \; p_2 = 4 - 10 = -6, \; p_3 = 4 - 10 = -6$

$p_4 = 6 - 10 = -4$, i.e., only player 1 pays, others get paid.

Surprisingly, this is a truthful mechanism.

Theorem: Groves mechanisms are DSIC.

Proof: Consider player i. Let $f^{eff}(\theta_i, \tilde{\underline{\theta}}_{-i}) = a$, and

$f^{eff}(\theta_i', \tilde{\underline{\theta}}_{-i}) = b$

by definition, $v_i(a, \theta_i) + \sum_{j \neq i} v_j(a, \tilde{\theta}_j)$

$$\geqslant v_i(b, \theta_i) + \sum_{j \neq i} v_j(b, \tilde{\theta}_j) \quad \cdots\cdots \; (1)$$

utility of player i when he reports θ_i

$$v_i(f^{eff}(\theta_i, \tilde{\underline{\theta}}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\underline{\theta}}_{-i})$$

$$= v_i(\underbrace{f^{eff}(\theta_i, \tilde{\underline{\theta}}_{-i})}_{a}, \theta_i) - h_i(\tilde{\underline{\theta}}_{-i}) + \sum_{j \neq i} v_j(\underbrace{f^{eff}(\theta_i, \tilde{\underline{\theta}}_{-i})}_{a}, \tilde{\theta}_j)$$

$$\geqslant v_i(\underbrace{f^{eff}(\theta_i', \tilde{\underline{\theta}}_{-i})}_{b}, \theta_i) \underbrace{- h_i(\tilde{\underline{\theta}}_{-i}) + \sum_{j \neq i} v_j(\underbrace{f^{eff}(\theta_i', \tilde{\underline{\theta}}_{-i})}_{b}, \tilde{\theta}_j)}_{= p_i(\theta_i', \tilde{\underline{\theta}}_{-i})}$$

$$= v_i\left(f^{eff}(\theta_i', \tilde{\underline{\theta}}_{-i}), \theta_i\right) - p_i(\theta_i', \tilde{\underline{\theta}}_{-i}).$$

Since player i was arbitrary, this holds for all $i \in N$.

Hence the claim.