VCG mechanism in combinatorial auctions

$M = \{1, \dots, m\}$: set of objects

$\Omega = 2^M := \{S : S \subseteq M\}$: set of bundles

$\theta_i : \Omega \to \mathbb{R}$: type/value of agent i

we assume $\theta_i(S) \geq 0 \ \forall S \in \Omega$, objects are "goods"

An allocation in this case is a partition of the objects

$a = \{a_0, a_1, a_2, \dots, a_n\}$, $a_i \in \Omega$, $a_i \cap a_j = \phi$ if $i \neq j$

$\bigcup_{i=0}^{n} a_i = M$. Let A be the set of all such allocations.

a_0 : set of unallocated objects.

Assume $\theta_i(\phi) = 0$

Also assume selfish valuations, i.e., $\theta_i(a) = \theta_i(a_i)$

agent i's valuation does NOT depend on the allocations to others.

Claim: In the allocation of goods, the VCG payment for agent, that gets no object in the efficient allocation, is zero.

Proof sketch: $a^* \in \underset{a \in A}{\arg\max} \sum_{j \in N} \theta_j(a)$, $a_i^* = \phi$

$$a_{-i}^* \in \underset{a \in A}{\arg\max} \sum_{j \in N \setminus \{i\}} \theta_j(a)$$

we know, $p_i^{VCG}(\theta) \geq 0$, also $p_i^{VCG}(\theta) = \sum_{j \neq i} \theta_j(a_{-i}^*) - \sum_{j \neq i} \theta_j(a^*)$

[add $\theta_i(a^*_{-i}) = 0$ and subtract $\theta_i(a^*) = \theta_i(a^*_i) = 0$]

$$= \sum_{j \in N} \theta_j(a^*_{-i}) - \sum_{j \in N} \theta_j(a^*) \leqslant 0$$ [a^* maximizes this sum by definition]

Hence $p_i^{VCG}(\theta) = 0$.

Defn: (Individual Rationality)

A mechanism $(f, \underline{p})$ is individually rational if

$$v_i(f(\theta), \theta_i) - p_i(\theta) \geqslant 0, \ \forall \theta \in \Theta, \ \forall i \in N.$$

Claim: In the allocation of goods, VCG mechanism is individually rational.

Proof sketch: $\theta_i(a^*) - p_i^{VCG}(\theta)$

$$= \theta_i(a^*) - \left(\sum_{j \neq i} \theta_j(a^*_{-i}) - \sum_{j \neq i} \theta_j(a^*)\right)$$

$$= \sum_{j \in N} \theta_j(a^*) - \sum_{j \neq i} \theta_j(a^*_{-i}) - \theta_i(a^*_{-i}) + \theta_i(a^*_{-i})$$

$$= \underbrace{\sum_{j \in N} \theta_j(a^*) - \sum_{j \in N} \theta_j(a^*_{-i})}_{\geqslant 0, \text{ by defn. of } a^*} + \underbrace{\theta_i(a^*_{-i})}_{\geqslant 0} \quad \geqslant 0$$