Mechanism design for selling a single indivisible object

Motivation: simplest yet elegant results

Setup: type set of agent i : $T_i \subseteq \mathbb{R}$

$t_i \in T_i$ denotes the value of agent i if she wins the object

An allocation a is a vector of length n that represents the probability of winning the object by the respective agent. Hence,

Set of allocations: $\Delta A = \{ a \in [0,1]^n : \sum_{i=1}^{n} a_i = 1 \}$

Allocation rule: $f : T_1 \times T_2 \times \cdots \times T_n \to \Delta A$

valuation: $v_i(a, t_i) = a_i \cdot t_i$ (product form) — expected valuation

Hence, $f_i(t_i, t_{-i})$ is the probability of winning the object for agent i when the type profile is (t_i, t_{-i}).

Recall: Vickrey / Second-price auction: type is v_i.

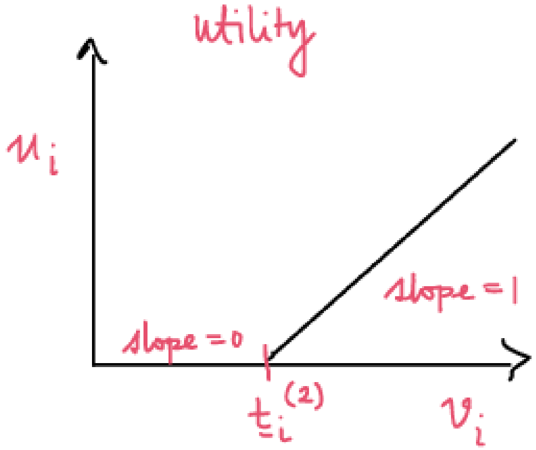
define $t_{-i}^{(2)} = \max_{j \neq i} \{v_j\}$

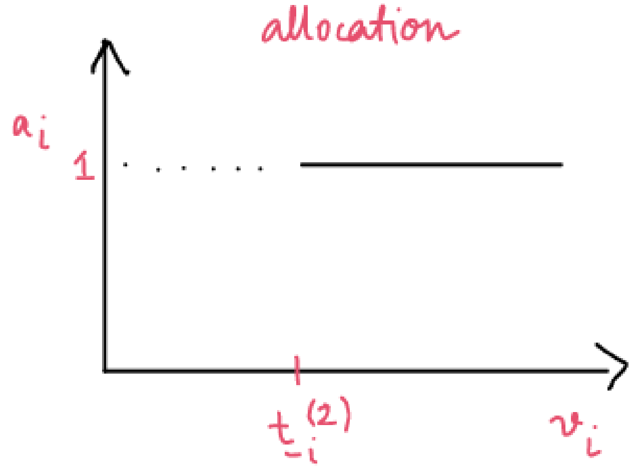
agent i wins if $v_i > t_{-i}^{(2)}$, loses if $v_i < t_{-i}^{(2)}$

a tie-breaking rule decides if equality.

Since, payment is $t_{-i}^{(2)}$ if i is the winner, the utility is zero in case of a tie.

$$u_i = \begin{cases} 0 & \text{if } v_i \leq t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & \text{if } v_i > t_{-i}^{(2)} \end{cases}$$





Observations:

1. utility is convex, derivative is zero if $v_i < \underline{t}_i^{(2)}$ and 1 if $v_i > \underline{t}_i^{(2)}$ – not differentiable at $v_i = \underline{t}_i^{(2)}$.
2. Whenever differentiable, it coincides with the allocation probability.

Known facts from convex analysis (see, e.g., Rockafeller (1980))

Fact 1: Convex functions are continuous in the interior of its domain. Jumps can occur only at the boundaries.

Fact 2: Convex functions are differentiable "almost everywhere".
The points where the function is not differentiable form a countable set (see the example before) – has measure zero.

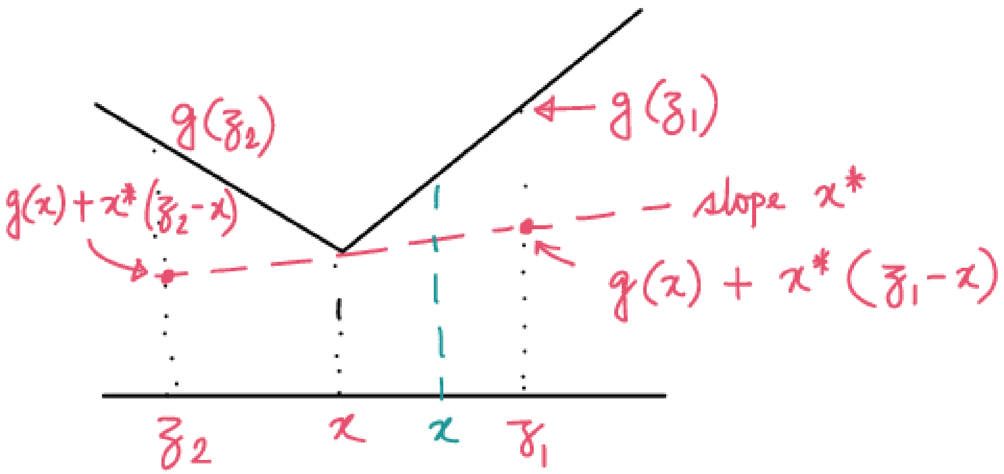
Recall: A function $g : I \to \mathbb{R}$ (where I is an interval) is convex if for every $x, y \in I$ and $\lambda \in [0,1]$

$$\lambda g(x) + (1-\lambda) g(y) \geq g(\lambda x + (1-\lambda) y).$$

If g is differentiable at $x \in I$, we denote the derivative by $g'(x)$. The following definition extends the idea of gradient

Defn: For any $x \in I$, x^* is a subgradient of g at x if

$$g(z) \geq g(x) + x^*(z-x) \quad \forall z \in I.$$



many possible x^* for x.

only one x^* possible for x

Few standard results (proofs: any standard text on convex analysis)

Lemma 1: Let $g: I \to \mathbb{R}$ be a convex function. Suppose x is in the interior of I and g is differentiable at x. Then $g'(x)$ is the unique subgradient of g.

Lemma 2: Let $g: I \to \mathbb{R}$ be a convex function. Then for every $x \in I$ a subgradient of g at x exists.

Fact 3: Let $I' \subseteq I$ be the set of points where g is differentiable. The set $I \setminus I'$ is of measure zero. The set of subgradients at a point forms a convex set.

Define $$g'_+(x) = \lim_{\substack{z \to x \\ z \in I', z > x}} g'(z), \qquad g'_-(x) = \lim_{\substack{z \to x \\ z \in I', z < x}} g'(z)$$

Fact 4: The set of subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$

We will denote the set of subgradients of g at $x \in I$ as $\partial g(x)$

Lemma 1 says $\partial g(x) = \{g'(x)\}$ $\forall x \in I'$.

Lemma 2 says that $\partial g(x) \neq \emptyset$ $\forall x \in I$.

Lemma 3: Let $g: I \to \mathbb{R}$ be a convex function. Let $\phi: I \to \mathbb{R}$ be a subgradient function, i.e., $\phi(z) \in \delta g(z)$ $\forall z \in I$. Then for all $x, y \in I$ s.t. $x > y$, we have $\phi(x) \geq \phi(y)$.

$\phi(z)$ picks one value at every z (even if subgradients can be many)

This result says that subgradient functions are monotone non-decreasing.

Lemma 4: Let $g: I \to \mathbb{R}$ be a convex function. Then for any $x, y \in I$,

$$g(x) = g(y) + \int_y^x \phi(z)\, dz,$$

where $\phi: I \to \mathbb{R}$ is s.t. $\phi(z) \in \delta g(z)$, $\forall z \in I$.