Monotonicity and Myerson's lemma

Defn: An allocation rule is non-decreasing if for every agent $i \in N$ and $\underline{t}_{-i} \in \underline{T}_{-i}$ we have $f_i(t_i, \underline{t}_{-i}) \geq f_i(s_i, \underline{t}_{-i}), \forall s_i, t_i \in T_i, t_i > s_i$

Holding the types of other agents fixed, the probability of allocation never decreases with valuation.

Theorem (Myerson 1981)

Suppose $T_i = [0, b_i]$ $\forall i \in N$, and the valuations are in the product form. An allocation rule $f: T \to \Delta A$ and a payment rule $(p_1, \dots, p_n)$ are DSIC iff

1. f is non-decreasing, and
2. payments are given by

$$p_i(t_i, \underline{t}_{-i}) = p_i(0, \underline{t}_{-i}) + t_i f_i(t_i, \underline{t}_{-i}) - \int_0^{t_i} f_i(x, \underline{t}_{-i})\, dx.$$

$$\forall t_i \in T_i, \forall \underline{t}_{-i} \in \underline{T}_{-i}, \forall i \in N.$$

Remark: difference with the Roberts' theorem: Roberts' result gives a functional form, while Myerson's result is a more implicit property. Sometimes function forms help answering questions in a more direct manner.

Proof: $(\Rightarrow)$ given that $(f, \underline{p})$ is DSIC.

Utility of agent i

$$u_i(t_i, \underline{t}_{-i}) = t_i f_i(t_i, \underline{t}_{-i}) - p_i(t_i, \underline{t}_{-i}), \text{ and}$$

$$u_i(s_i, \underline{t}_{-i}) = s_i f_i(s_i, \underline{t}_{-i}) - p_i(s_i, \underline{t}_{-i}).$$

Since $(f, \underline{p})$ is DSIC,

$$u_i(t_i, \underline{t}_i) = t_i f(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)$$
$$\geq t_i f(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)$$
$$= s_i f_i(s_i, \underline{t}_i) + f_i(s_i, \underline{t}_i)(t_i - s_i) - p_i(s_i, \underline{t}_i)$$
$$= u_i(s_i, \underline{t}_i) + f_i(s_i, \underline{t}_i)(t_i - s_i) \quad \cdots\cdots (1)$$

fixing $\underline{t}_i$, define $g(t_i) = u_i(t_i, \underline{t}_i), \phi(t_i) = f_i(t_i, \underline{t}_i)$.

Hence, Eq (1) can be written as

$$g(t_i) \geq g(s_i) + \phi(s_i)(t_i - s_i)$$

$\Rightarrow$ $\phi(s_i)$ is a subgradient of g at s_i. $\cdots\cdots$ (2)

Next, need to show: g is convex.

pick $x_i, z_i \in T_i$, define $y_i = \lambda x_i + (1-\lambda) z_i$, $\lambda \in [0,1]$.

DSIC implies

$$g(x_i) \geq g(y_i) + \phi(y_i)(x_i - y_i), \quad \text{and}$$
$$g(z_i) \geq g(y_i) + \phi(y_i)(z_i - y_i)$$
$$\Rightarrow \lambda g(x_i) + (1-\lambda) g(z_i) \geq g(y_i) + \phi(y_i)[\underbrace{\lambda x_i + (1-\lambda) z_i - y_i}_{=0}]$$
$$= g(\lambda x_i + (1-\lambda) z_i)$$

$\Rightarrow$ g is convex. $\cdots\cdots$ (3)

Apply lemmas 3 and 4

Lemma 3 $\Rightarrow$ ϕ is non-decreasing, i.e., $f_i(\cdot, \underline{t}_i)$ is non-decreasing

$\Rightarrow$ Part (1) is proved.

Lemma 4 $\Rightarrow$ $g(t_i) = g(0) + \int_0^{t_i} \phi(x)\, dx$

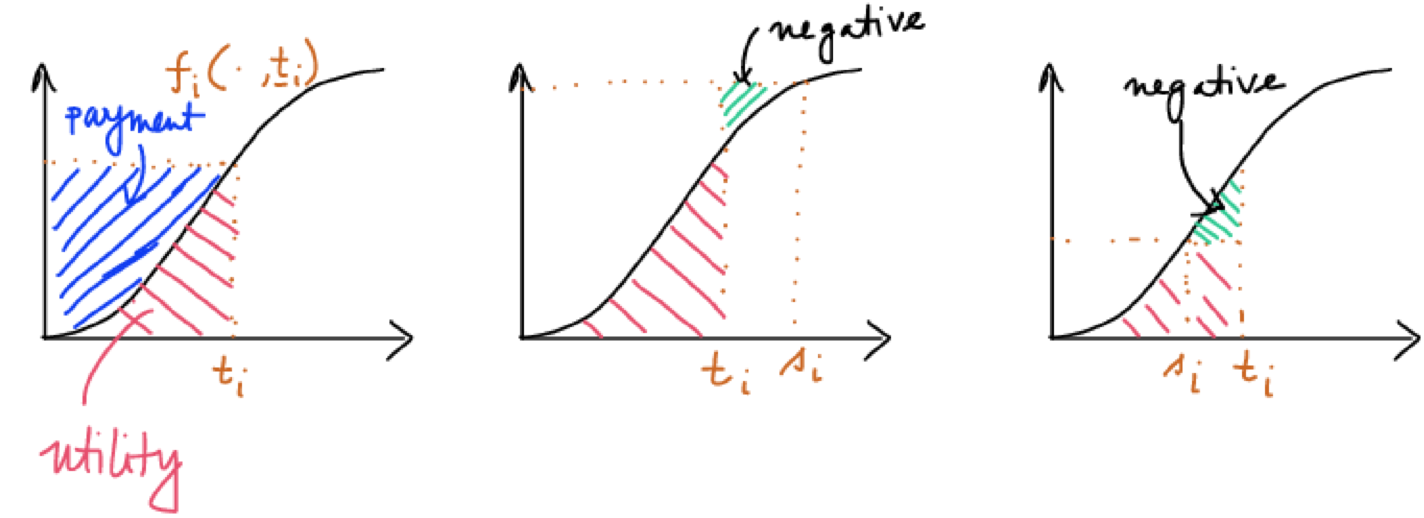
$$\Rightarrow u_i(t_i, \underline{t}_i) = u_i(0, \underline{t}_i) + \int_0^{t_i} f_i(x, \underline{t}_i)\, dx$$

$$\Rightarrow t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i) = -p_i(0, \underline{t}_i) + \int_0^{t_i} f_i(x, \underline{t}_i)\, dx$$

$$\Rightarrow p_i(t_i, \underline{t}_i) = p_i(0, \underline{t}_i) + t_i f_i(t_i, \underline{t}_i) - \int_0^{t_i} f_i(x, \underline{t}_i)\, dx.$$

($\Leftarrow$) Given: f is non-decreasing and payment formula.

proof by pictures – assume $p_i(0, \underline{t}_i) = 0$



$$\left[t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)\right] - \left[t_i f_i(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)\right]$$

$$= (s_i - t_i) f_i(s_i, \underline{t}_i) + \int_{s_i}^{t_i} f_i(x, \underline{t}_i)\, dx \geq 0$$

Corollary: An allocation rule in single object allocation setting is implementable in dominant strategies if it is non-decreasing.