Examples of some single object allocation mechanisms

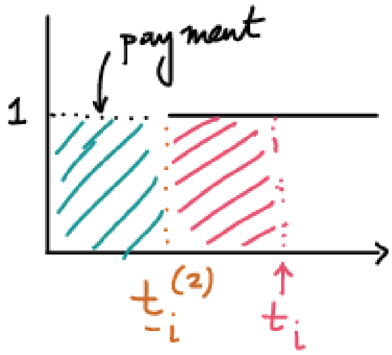
① Constant allocation rule – non-decreasing, payment = constant (e.g., 0)

② Dictatorial – give the object only to the dictator – non-decreasing, payment = constant / zero.

③ Second price auction

$$\underbrace{p_i(0,t_{-i})}_{=0} + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})\, dx .$$

step function



④ Efficient allocation with a reserve price is also non decreasing.
If the highest value is below a reserve price r, nobody gets the object. Otherwise, the item goes to the highest bidder.
allocated to i if $v_i > \max\{t_{-i}^{(2)}, r\}$. payment $= \max\{t_{-i}^{(2)}, r\}$

⑤ Not so common allocation rule : $N = \{1,2\}$, $A = \{a_0, a_1, a_2\}$

a_0: unsold, a_1: given to 1

Given a type profile $t = (t_1, t_2)$, the seller computes
$U(t) = \max\{2, t_1^2, t_2^3\}$ – select a_0, a_1, a_2 depending on which of the three expressions is the maxima – break ties in favor of $0 > 1 > 2$.

Player 1 gets the object if $t_2 > \sqrt{\max\{2, t_2^3\}}$

Player 2 gets the object if $t_3 > \sqrt[3]{\max\{2, t_1^2\}}$

both monotone.

Individual Rationality

Defn: A mechanism $(f, \underline{p})$ is *ex post individually rational* if

$$t_i f_i(t_i, \underline{t}_{-i}) - p_i(t_i, \underline{t}_{-i}) \geq 0, \ \forall t_i \in T_i, \forall \underline{t}_{-i} \in \underline{T}_{-i}, \forall i \in N.$$

Ex-post: even after all agents have revealed their types, participating is weakly preferred.

Lemma: In the single object allocation setting, consider a DSIC mechanism $(f, \underline{p})$.

1. It is IR iff $\forall i \in N$ and $\forall \underline{t}_{-i} \in \underline{T}_{-i}$, $p_i(0, \underline{t}_{-i}) \leq 0$.
2. It is IR and satisfies no subsidy, i.e., $p_i(t_i, \underline{t}_{-i}) \geq 0$, $\forall t_i, \underline{t}_{-i}$ $\forall i \in N$ iff $\forall i \in N$, $\underline{t}_{-i} \in \underline{T}_{-i}$, $p_i(0, \underline{t}_{-i}) = 0$.

Proof: (Part 1) Suppose $(f, \underline{p})$ is IR, Then $0 - p_i(0, \underline{t}_{-i}) \geq 0$

hence $p_i(0, \underline{t}_{-i}) \leq 0$.

Conversely, if $p_i(0, \underline{t}_{-i}) \leq 0$, then the payoff of i is

$$t_i f_i(t_i, \underline{t}_{-i}) - p_i(t_i, \underline{t}_{-i})$$

$$= t_i f_i(t_i, \underline{t}_{-i}) \underbrace{- p_i(0, \underline{t}_{-i})}_{\geq 0} - t_i f_i(t_i, \underline{t}_{-i}) + \int_0^{t_i} f_i(x, \underline{t}_{-i})\, dx \geq 0$$

(Part 2): IR $\Rightarrow p_i(0, \underline{t}_{-i}) \leq 0$, if $p_i(t_i, \underline{t}_{-i}) \geq 0 \ \forall t_i \Rightarrow$ $p_i(0, \underline{t}_{-i}) = 0$.

Clearly if $p_i(0, \underline{t}_{-i}) = 0 \Rightarrow (f, \underline{p})$ is IR and no-subsidy.

Some non-Vickrey auctions – focus: budget balance

(1) The object goes to the highest bidder, but the payment is such that everyone is compensated some amount.

- highest and second highest bidders are compensated $\frac{1}{n}$ of the third highest bid. $p_1(0, \underline{t}_{-1}) = p_2(0, \underline{t}_{-2}) = -\frac{1}{n} t_3$
- everyone else receives $\frac{1}{n}$ of the second highest bid

$$p_i(0, \underline{t}_{-i}) = -\frac{1}{n} \text{ second highest in } \{t_j, j \neq i\}$$

WLOG $t_1 > t_2 > \cdots > t_n$

Agent 1 pays $= -\frac{1}{n} t_3 + t_1 - \int_0^{t_1} f_1(x, \underline{t}_{-1})\, dx = -\frac{1}{n} t_3 + t_2$

2 pays $= -\frac{1}{n} t_3$, all others $= -\frac{1}{n} t_2$

total payments $= -\frac{1}{n} t_3 + t_2 - \frac{1}{n} t_3 - \frac{n-2}{n} t_2 = \frac{2}{n}(t_2 - t_3)$

tends to 0 for large n.

deterministic mechanism that redistributes the money.

(2) Allocate the object w.p. $\left(1-\frac{1}{n}\right)$ to the highest bidder
w.p. $\frac{1}{n}$ to the second highest bidder

$$p_i(0, \underline{t}_{-i}) = -\frac{1}{n} \text{ second highest bid in } \{t_j, j \neq i\}$$

1 pays $= -\frac{1}{n} t_3 + \left(1-\frac{1}{n}\right) t_1 - \frac{1}{n}(t_2 - t_3) - \left(1-\frac{1}{n}\right)(t_1 - t_2)$

$= \left(1-\frac{2}{n}\right) t_2$

2 pays $= -\frac{1}{n} t_3 + \frac{1}{n} t_2 - \frac{1}{n}(t_2 - t_3) = 0$

all others $= -\frac{1}{n} t_2$. Together $= 0$.