How to maximize the revenue earned by the auctioneer?

maximize w.r.t. what knowledge of the auctioneer? — The common prior distribution over the types.

Accordingly, the notions of incentive compatibility and individual rationality have to change.

Bayesian Incentive Compatibility

$T_i = [0, b_i]$, common prior G over $T = \bigtimes_{i=1}^{n} T_i$ — g denotes the density

$G_{-i}(\underline{s}_{-i} | s_i)$ is the conditional distribution over s_{-i}, given i's type is s_i.

similarly $g_{-i}(\underline{s}_{-i} | s_i)$ is derived via Bayes rule from g.

Every mechanism $(f, p_1, \dots, p_n)$ induces an expected allocation and payment rule $(\alpha, \underline{\pi})$

$$\alpha_i(\overset{\text{reported}}{s_i} | \overset{\text{true}}{t_i}) = \int_{\underline{s}_{-i} \in \underline{T}_{-i}} \underbrace{f_i(s_i, \underline{s}_{-i})}_{\text{probabilistic allocation}} \underbrace{g_{-i}(\underline{s}_{-i} | t_i)}_{\substack{\text{types of other agents} \\ \text{common prior}}} d\underline{s}_{-i}$$

two levels of expectation

expected payment

$$\pi_i(s_i | t_i) = \int_{\underline{s}_{-i} \in \underline{T}_{-i}} p_i(s_i, \underline{s}_{-i}) \, g_{-i}(\underline{s}_{-i} | t_i) \, d\underline{s}_{-i}$$

Expected utility of agent i

$$t_i \alpha_i(t_i | t_i) - \pi_i(t_i | t_i)$$

<u>Defn</u>: A mechanism $(f, \underline{p})$ is Bayesian incentive compatible (BIC) if $\forall i \in N$, $\forall s_i, t_i \in T_i$

$$t_i \alpha_i(t_i | t_i) - \pi_i(t_i | t_i) \geqslant t_i \alpha_i(s_i | t_i) - \pi_i(s_i | t_i).$$

Similarly, f is Bayesian implementable if $\exists\ \underline{p}$ s.t. $(f, \underline{p})$ is BIC

Independence and Characterization of BIC mechanisms

Assume that the priors are independent, i.e., agent i's value is drawn from a distribution G_i (density g_i) independently from other agents.

$$G(s_1, s_2, \ldots, s_n) = \prod_{i \in N} G_i(s_i)$$

$$G(\underline{s}_{-i} \mid t_i) = \prod_{j \neq i} G_j(s_j)$$

We will use the shorthand $\alpha(t_i) = \alpha(t_i \mid t_i)$

__Defn__: An allocation rule is non-decreasing in expectation (NDE) if $\forall i \in N$, $\forall s_i, t_i \in T_i$ with $s_i < t_i$ we have $\alpha_i(s_i) \leq \alpha_i(t_i)$.

Note: The rules that are non-decreasing (defined before) are always NDE. But there can be more rules that are NDE.

Characterization of BIC rules

__Theorem__: A mechanism $(f, \underline{p})$ in the independent prior setting is BIC iff ① f is NDE, and

② p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx$

$\forall t_i \in T_i$, $\forall i \in N$.

Remark: Bayesian version of the earlier theorem

Proof: in similar lines as before [exercise]

An allocation rule may be NDE but not non-decreasing.

t_2

				1
			1	
			1	1
		1		1

t_1

all 5 types are equally likely

$\alpha_1(t_1)$ and $\alpha_2(t_2)$ are monotone but $f(t_1, t_2)$ is not.

As we are in the Bayesian setting now, we can define an analog of individual rationality

Defn: A mechanism $(f, \underline{p})$ is interim individually rational (IIR) if for every bidder $i \in N$, we have

$$t_i \alpha_i(t_i) - \pi_i(t_i) \geqslant 0 \quad \forall\, t_i \in T_i .$$

Lemma: A mechanism $(f, \underline{p})$ is BIC and IIR iff

1. f is NDE,
2. p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx$
$\forall t_i \in T_i, \forall i \in N.$
3. $\forall\, i \in N, \ \pi_i(0) \leqslant 0.$

Proof sketch: 1 and 2 uniquely identify a BIC mechanism. So, the proof requires to show that IIR along with 1 and 2 are equivalent to 3

($\Rightarrow$) apply IIR at $t_i = 0$ on 2 and get $\pi_i(0) \leqslant 0$

($\Leftarrow$) $t_i \alpha_i(t_i) - \pi_i(t_i) = -\pi_i(0) + \int_0^{t_i} \alpha_i(s_i)\, ds_i \geqslant 0$ if $\pi_i(0) \leqslant 0$