Optimal mechanism design for a single agent

Motivation: analyze a simpler problem to understand the problem of revenue maximization. Will generalize later to multiple agents.

<u>Setup</u>: Type set $T = [0, \beta]$. Mechanism (f, p)

$$f : [0,\beta] \to [0,1], \quad p : [0,\beta] \to \mathbb{R}$$

- Incentive compatibility [BIC and DSIC equivalent]

$$t f(t) - p(t) \geq t f(s) - p(s), \quad \forall t, s \in T.$$

- Individual rationality [IR and IIR equivalent]

$$t f(t) - p(t) \geq 0, \quad \forall t \in T.$$

The expected revenue earned by a mechanism M is given by

$$\pi^M := \int_0^\beta p(t) g(t)\, dt$$

We need to find a mechanism M^* in the class of all IC and IR mechanisms s.t. $\pi^{M^*} \geq \pi^M, \ \forall M$.

We will call M^* the optimal mechanism.

Q: What is the structure of an optimal mechanism?

Consider an IC and IR mechanism $(f, p) \equiv M$

By the characterization theorems and lemmas, we know

$$p(t) = p(0) + t f(t) - \int_0^t f(x)\, dx \quad \text{[IC]}$$

$$p(0) \leq 0 \quad \text{[IR]}$$

Since we want to maximize revenue, $p(0) = 0$.

Hence, the payment formula is

$$p(t) = t f(t) - \int_0^t f(x)\,dx$$

Note: in optimal mechanism, payment is completely given once the allocation is fixed. Hence, we need to optimize only over one variable.

Expected revenue: $\pi^f = \int_0^\beta p(t)\, g(t)\,dt$

$$= \int_0^\beta \left(t f(t) - \int_0^t f(x)\,dx\right) g(t)\,dt$$

Need to maximize this wrt f.

Lemma: For any implementable allocation rule f, we have

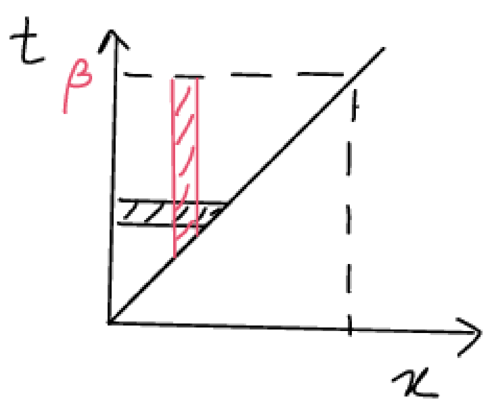
$$\pi^f = \int_0^\beta \left(t - \frac{1-G(t)}{g(t)}\right) g(t)\,dt .$$

Proof: $\pi^f = \int_0^\beta \left(t f(t) - \int_0^t f(x)\,dx\right) g(t)\,dt$

$$= \int_0^\beta t f(t)\, g(t)\,dt - \int_0^\beta \int_0^t f(x)\,dx \; g(t)\,dt$$

$$= \int_0^\beta t f(t)\, g(t)\,dt - \int_0^\beta \int_x^\beta g(t)\,dt \; f(x)\,dx$$

[standard limit switching]



$$= \int_0^\beta t f(t)\, g(t)\,dt - \int_0^\beta \int_t^\beta g(x)\,dx \; f(t)\,dt$$

$$= \int_0^\beta \left[t f(t) g(t) - (1 - G(t)) f(t) \right] dt$$

$$= \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t)\, dt . \qquad \square$$

Hence the optimal mechanism finding problem reduces to

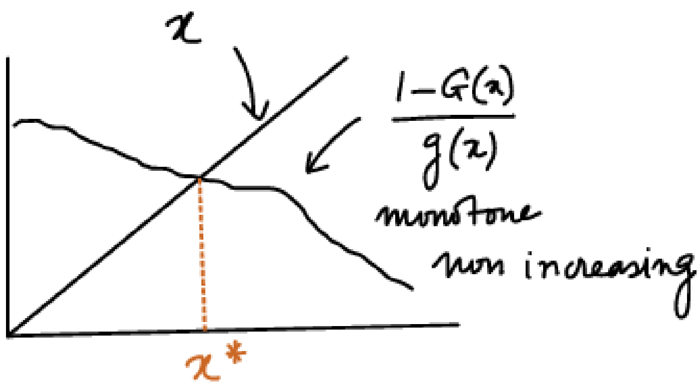
OPT1: $\displaystyle \max_{f : f \text{ is nondecreasing}} \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t)\, dt$

Assumption: G satisfies the monotone hazard rate condition (MHR), i.e., $\dfrac{g(x)}{1 - G(x)}$ is non decreasing in x.

Standard distributions like uniform and exponential satisfy MHR condition.

Fact: If G satisfies MHR condition, there is a unique solution to

$$x = \frac{1 - G(x)}{g(x)} .$$



Intuition:

Let x^* be the unique solution of this equation

Hence, $w(x) = x - \dfrac{1 - G(x)}{g(x)}$ is zero at x^*

$w(x) > 0 \;\; \forall x > x^*$ and $< 0 \;\; \forall x < x^*$.

The unrestricted solution to OPT1 is therefore

$$f(t) = \begin{cases} 0 & \text{if } t < x^* \\ 1 & \text{if } t > x^* \\ \alpha & \text{if } t = x^*, \; \alpha = [0,1] \end{cases} \quad \text{..........} \; 1$$

But this f is non-decreasing, therefore it is the optimal solution of OPT1.

Theorem: A mechanism (f, p) under the MHR condition is optimal iff ① f is given by eqn. ① where x^* is the unique solution of $x = \frac{1-G(x)}{g(x)}$, and

② For all $t \in T$, $p(t) = \begin{cases} x^* & \text{if } t \geq x^* \\ 0 & \text{ow} \end{cases}$