Optimal mechanism design for multiple agents

In this context, we will call a mechanism optimal if it is BIC and IIR and maximizes revenue.

By previous results, this reduces to
1. f_i's are NDE, $\forall i \in N$,
2. $\pi_i(t_i)$ has a specific formula and $\pi_i(0) = 0$.

The expected payment made by agent i is

$$\int_{T_i} \pi_i(t_i)\, g_i(t_i)\, dt_i \quad ; \quad T_i = [0, b_i]$$

in a way similar to the earlier exercise, simplify to the following

$$\int_0^{b_i} w_i(t_i)\, g_i(t_i)\, \underbrace{\alpha_i(t_i)}_{= \int_{T_{-i}} f_i(t_i, t_{-i})\, g_{-i}(t_{-i})\, dt_{-i}}\, dt_i \qquad w_i(t_i) = t_i - \frac{1 - G_i(t_i)}{g_i(t_i)}$$

also called virtual valuation of player i

$$= \int_T w_i(t_i)\, f_i(t)\, g(t)\, dt$$

Hence, the total revenue generated by all players is

$$\sum_{i \in N} \int_T w_i(t_i)\, f_i(t)\, g(t)\, dt$$

$$= \int_T \left(\sum_{i \in N} w_i(t_i)\, f_i(t) \right) g(t)\, dt$$

expected total virtual valuation

Hence the optimal mechanism design problem reduces to

$$\max \int_T \left(\sum_{i \in N} w_i(t_i)\, f_i(t) \right) g(t)\, dt \quad , \text{ s.t. } f \text{ is NDE.}$$

As before, we attempt to solve the unconstrained optimization problem.

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j) \ \forall j \quad \text{(sold)} \\ 0 & \text{ow} \end{cases} \qquad \text{①}$$

$$f_i(t) = 0, \forall i \in N, \text{ if } w_i(t_i) < 0 \ \forall i \in N \quad \text{(unsold)}$$

But it can lead to a situation where f is not NDE

(for an example, see Myerson (1981): "Optimal Auction Design" – The example is such that the following condition is violated)

Defn: A virtual valuation w_i is regular if $\forall s_i, t_i \in T_i$ with $s_i < t_i$, it holds that $w_i(s_i) < w_i(t_i)$.

This condition is weaker than the MHR condition as MHR implies regularity.

Lemma: Suppose every agent's valuations are regular. The allocation rule of the optimal mechanism is same as the solution of the unconstrained problem.

Proof sketch: The solution is as given in eqn. ①.

Regularity ensures that $w_i(t_i) > w_i(s_i) \ \forall s_i < t_i$

Then the optimal allocation rule also satisfies

$$f_i(t_i, \underline{t}_i) \geqslant f_i(s_i, \underline{t}_i) \ \forall \underline{t}_i \in \underline{T}_i, \ \forall s_i < t_i.$$

i.e., f_i is non-decreasing (hence NDE).