Bayesian Equilibrium Existence Proof

Theorem 1. *Every finite Bayesian game has a Bayesian equilibrium.*

Proof. We prove this by converting the Bayesian game into a complete information normal form game as follows. Consider each type of every player as an independent player. Thus, the expanded player set is given by

$$\tilde{N} = \cup_{i \in N} \Theta_i = \{\theta_1^1, \theta_1^2, \dots, \theta_1^{|\Theta_1|}, \theta_2^1, \theta_2^2, \dots, \theta_2^{|\Theta_2|}, \dots, \theta_n^1, \theta_n^2, \dots, \theta_n^{|\Theta_n|}\}$$

In the rest of the proof, we will consider two players, each having two types. However, the same proof extends for any finite number of player and their types.
The utility of player θ_1^1 is given by

$$\begin{aligned}
&\tilde{u}_{\theta_1^1}(a_{\theta_1^1}, a_{\theta_1^2}, a_{\theta_2^1}, a_{\theta_2^2}) \\
&= P(\theta_2^1|\theta_1^1) u_1(a_{\theta_1^1}, a_{\theta_2^1}; \theta_1^1, \theta_2^1) + P(\theta_2^2|\theta_1^1) u_1(a_{\theta_1^1}, a_{\theta_2^2}; \theta_1^1, \theta_2^2) \qquad (1)\\
&\quad \text{where } a_{\theta_i^j} = a_i(\theta_i^j) \text{ and } u_i \text{ is the utility function of the original Bayesian game.}
\end{aligned}$$

The set of actions in this expanded game of a player θ_i^j is the same set that player i had under the type θ_i^j in the original game, which is given by A_i. Consider a mixed strategy profile $(\sigma_{\theta_1^1}, \sigma_{\theta_1^2}, \sigma_{\theta_2^1}, \sigma_{\theta_2^2})$ in this new game. The utility is given by

$$\begin{aligned}
&\tilde{u}_{\theta_1^1}(\sigma_{\theta_1^1}, \sigma_{\theta_1^2}, \sigma_{\theta_2^1}, \sigma_{\theta_2^2}) \\
&= \sum_{a_{\theta_2^2} \in A_2} \sum_{a_{\theta_2^1} \in A_2} \sum_{a_{\theta_1^2} \in A_1} \sum_{a_{\theta_1^1} \in A_1} \sigma_{\theta_1^1}(a_{\theta_1^1}) \sigma_{\theta_1^2}(a_{\theta_1^2}) \sigma_{\theta_2^1}(a_{\theta_2^1}) \sigma_{\theta_2^2}(a_{\theta_2^2}) \;\; \tilde{u}_{\theta_1^1}(a_{\theta_1^1}, a_{\theta_1^2}, a_{\theta_2^1}, a_{\theta_2^2}) \\
&= \sum_{a_{\theta_2^1} \in A_2} P(\theta_2^1|\theta_1^1) \sigma_{\theta_2^1}(a_{\theta_2^1}) u_1(a_{\theta_1^1}, a_{\theta_2^1}; \theta_1^1, \theta_2^1) + \sum_{a_{\theta_2^2} \in A_2} P(\theta_2^2|\theta_1^1) \sigma_{\theta_2^2}(a_{\theta_2^2}) u_1(a_{\theta_1^1}, a_{\theta_2^2}; \theta_1^1, \theta_2^2) \\
&\quad \text{Replace } \sigma_{\theta_i^j}(a_{\theta_i^j}) =: \sigma_i(\theta_i^j, a_{\theta_i^j}), \text{ to get} \\
&= \sum_{\theta_2 \in \Theta_2} P(\theta_2|\theta_1^1) u_1(\sigma_1, \sigma_2|\theta_1^1)
\end{aligned}$$

The first equality comes by definition. In the second equality, we substitute the expression of the utility from Eq. (1) and simplify by summing the irrelevant $\sigma_{\theta_i^j}(a_{\theta_i^j})$ terms to 1. The substitution after that step is by definition of σ. In the final step, we equate the previous step by combining the two conditional probability terms w.r.t. the original Bayesian game.
Hence, a mixed strategy profile $(\sigma_{\theta_1^1}, \sigma_{\theta_1^2}, \sigma_{\theta_2^1}, \sigma_{\theta_2^2})$ in the complete information game is a mixed strategy profile (σ_1, σ_2) in the original Bayesian game. By Nash's theorem, a mixed strategy

Nash equilibrium in a finite complete information game always exists. Hence, we conclude that a Bayesian equilibrium always exists in a finite Bayesian game. □