भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 3

Swaprava Nath

ज्ञानम् परमम् ध्येयम्
Knowledge is the supreme goal

Contents

A special class with certain nice **security** and **stability** properties

---

Definition (Two player zero-sum games)

A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$

Question

Why called **matrix** game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game

Player 2

	L	R
L	−1, 1	1, −1
R	1, −1	−1, 1

Player 1

$\implies$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

*Player 2's **maxmin** value is the **minmax** value of this matrix*

	L	R	maxmin
L	−1	1	−1
R	1	−1	−1
minmax	1	1	

Player 1

Another example

Player 2

	L	C	R
T	$3, -3$	$-5, 5$	$-2, 2$
M	$1, -1$	$4, -4$	$1, -1$
B	$6, -6$	$-3, 3$	$-5, 5$

Player 1

	L	C	R	maxmin
T	3	-5	-2	-5
M	1	4	1	1
B	6	-3	-5	-5
minmax	6	4	1	

Player 1

	L	R	maxmin
L	−1	1	−1
R	1	−1	−1
minmax	1	1	

Player 1

	L	C	R	maxmin
T	3	−5	−2	−5
M	1	4	1	1
B	6	−3	−5	−5
minmax	6	4	1	

Player 1

Question

What are the PSNEs for the above games?

Answer

Left: no PSNE; **Right:** (M,R)

Saddle point

Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

Question

What are the saddle points for the above games?

Answer

For the first example: no saddle point, for the second: (M,R)

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point iff it is a PSNE.

Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2), \forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \geqslant u_1(s_1, s_2^*), \forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \geqslant u_2(s_1^*, s_2), \forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE. $\square$

Consider **maxmin** and **minmax** values

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \qquad\qquad \textbf{maxmin}$$

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) \qquad\qquad \textbf{minmax}$$

Question

How are the maxmin and minmax values related?

Relationship of the security values

Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.

Proof.

$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \ \forall s_1, s_2, \qquad \text{definition of min}$$

$$\Rightarrow \max_{t_1 \in S_1} u(t_1, s_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \ \forall s_2 \in S_2 \qquad \text{RHS was dominated for each } s_1$$

$$\Rightarrow \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \qquad \text{RHS was a constant}$$

$\square$

Contents

	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
minmax	1	1	

Player 1

$$\overline{v} = 1 > -1 = \underline{v}$$
PSNE does not exist

	L	C	R	maxmin
T	3	-5	-2	-5
M	1	4	1	1
B	6	-3	-5	-5
minmax	6	4	1	

Player 1

$$\overline{v} = 1 = \underline{v}$$
PSNE exists

Define the following strategies

$$s_1^* \in \arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2), \qquad \textbf{maxmin strategy of player 1}$$

$$s_2^* \in \arg \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2), \qquad \textbf{minmax strategy of player 2}$$

Theorem

A game has a PSNE (equivalently, a saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively.

Observation: (s_1^*, s_2^*) is a PSNE

<div align="center">Proof</div>

($\implies$) i.e., if (s_1^*, s_2^*) is a PSNE $\implies \overline{v} = \underline{v} = u(s_1^*, s_2^*)$

Since (s_1^*, s_2^*) is a PSNE, $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$.

$$\begin{aligned}
\implies u(s_1^*, s_2^*) &\geqslant \max_{t_1 \in S_1} u(t_1, s_2^*) \\
&\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy} \\
&= \overline{v}
\end{aligned}$$

Similarly, using the same argument for player 2, we get $\underline{v} \geqslant u(s_1^*, s_2^*)$, since for player 2, utility $u_2 \equiv -u$

But $\overline{v} \geqslant \underline{v}$ (from the previous lemma), hence

$$u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*)$$
$$\implies u(s_1^*, s_2^*) = \overline{v} = \underline{v}$$

<div style="text-align:center">Proof (contd.)</div>

$(\Longleftarrow)$ i.e. $\bar{v} = \underline{v} = u(s_1^*, s_2^*) \implies (s_1^*, s_2^*)$ is a PSNE

$$u(s_1^*, s_2) \geqslant \min_{t_2 \in S_2} u(s_1^*, t_2), \text{ by definition of min}$$
$$= \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \text{ since } s_1^* \text{ is the maxmin strategy for player 1}$$
$$= v \text{ (given)}$$

Similarly, we can show $u(s_1, s_2^*) \leqslant v, \ \forall s_1 \in S_1$
But $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE.

Contents

Mixed strategy: *probability distribution* over the set of strategies of that player

Player 2

	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$ L	$-1, 1$	$1, -1$
$\frac{1}{3}$ R	$1, -1$	$-1, 1$

Player 1

- Consider a finite set A, define

$$\Delta A = \{p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1\}$$

- Mixed strategy set of player 1: $\Delta S_1 = \Delta\{L, R\}$, $(\frac{2}{3}, \frac{1}{3}) \in \Delta S_1$

- **Notation**: σ_i is a mixed strategy of player i
- $\sigma_i \in \Delta S_i$, i.e. , $\sigma_i : S_i \to [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- We are discussing non-cooperative games, the players choose their strategies independently
- The joint probability of player 1 picking s_1 and player 2 picking $s_2 = \sigma_1(s_1)\sigma_2(s_2)$
- Utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \, u_i(s_1, s_2, \ldots, s_n)$$

- We are *overloading* u_i to denote the utility at *pure* and *mixed* strategies
- Utility at a mixed strategy is the **expectation** of the utilities at pure strategies; all the rules of expectation hold, e.g., linearity, conditional expectation, etc.

Player 2

	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$ L	$-1, 1$	$1, -1$
$\frac{1}{3}$ R	$1, -1$	$-1, 1$

Player 1

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1) + \frac{1}{3} \cdot \frac{4}{5} \cdot (1) + \frac{1}{3} \cdot \frac{1}{5} \cdot (-1)$$

Contents

Mixed Strategies Nash Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

A **mixed strategy Nash equilibrium (MSNE)** is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i \text{ and } \forall i \in N.$$

Question

Relation between **PSNE** and **MSNE**?

Answer

PSNE $\implies$ MSNE

Theorem

*A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, is an **MSNE** if and only if $\forall s_i \in S_i$ and $\forall i \in N$*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(s_i, \sigma_{-i}^*).$$

Proof.

$(\Rightarrow)$: The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE

$(\Leftarrow)$ Pick an arbitrary mixed strategy σ_i of player i

$$u_i(\sigma_i, \sigma_i^*) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \underbrace{\overset{\text{(given)}}{u_i(s_i, \sigma_{-i}^*)}}_{\leqslant u_i(\sigma_i^*, \sigma_{-i}^*)} u_i(s_i, \sigma_{-i}^*)$$

$$\leqslant \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(\sigma_i^*, \sigma_{-i}^*)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*) \cdot \sum \sigma_i(s_i) = u_i(\sigma_i^*, \sigma_{-i}^*)$$

Is the mixed strategy profile an **MSNE**?

Player 2

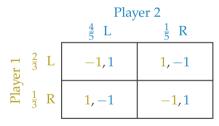|		$\frac{4}{5}$ L	$\frac{1}{5}$ R
Player 1	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
	$\frac{1}{3}$ R	$1, -1$	$-1, 1$

- To answer this, we need to show that there does not exist any better mixed strategy for the player
- Expected utility of player 2 from $L = 2/3 \cdot 1 + 1/3 \cdot (-1) = 1/3$
- Expected utility of player 2 from $R = 2/3 \cdot (-1) + 1/3 \cdot 1 = -1/3$

Is the mixed strategy profile an **MSNE**?

Player 2

	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$ L	$-1, 1$	$1, -1$
$\frac{1}{3}$ R	$1, -1$	$-1, 1$

Player 1

Player 2

	$\frac{1}{2}$ L	$\frac{1}{2}$ R
$\frac{1}{2}$ L	$-1, 1$	$1, -1$

er 1

Contents

Definition (Support of mixed strategy/probability distribution)

For mixed strategy σ_i, the subset of strategy set of i on which σ_i has a positive mass is called the **support** of σ_i and is denoted by $\delta(\sigma_i)$. Formally, $\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$.

Using the definition of support, here is a characterization of MSNE

Theorem

A mixed strategy profile is an MSNE iff [a] $\forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ *is identical* $\forall s_i \in \delta(\sigma_i^*)$,

2. $u_i(s_i, \sigma_{-i}^*) \geqslant u_i(s_i', \sigma_{-i}^*), \forall s_i \subseteq \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*).$

—————————————————————————
[a]This is a shorthand for 'if and only if'.

Consider Penalty Shoot Game

<div align="center">

Goalkeeper

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

Shooter

</div>

Case 1: support profile $(\{L\}, \{L\})$: for player 1 , $s_1' = R$ – violates condition 2

Case 2: support profile $(\{L, R\}, \{L\})$ – symmetric for the other case

For Player 1, the expected utility has to be the same for L and R - not possible – violates condition 1

Implication

Case 3: support profile $(\{L, R\}, \{L, R\})$: condition 2 is vacuously satisfied

For condition 1, let player 1 chooses L w.p. p and player 2 choose L w.p. q

For player 1:

$$u_1(L, (q, 1-q)) = u_1(R, (q, 1-q)) \Rightarrow (-1)q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q) \Rightarrow q = \frac{1}{2}$$
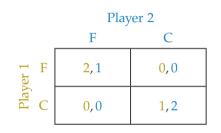
For player 2:

$$u_2((p, 1-p), L) = u_2((p, 1-p), R) \Rightarrow p = \frac{1}{2}$$

MSNE =

$$\left(\left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{2}, \frac{1}{2} \right) \right)$$

Player 2

	F	C
F	2, 1	0, 0
C	0, 0	1, 2

Player 1

Player 2

	F	C	D
F	2, 1	0, 0	1, 1
C	0, 0	1, 2	2, 0

Player 1

Contents

MSNE Characterization Theorem

Theorem

A mixed strategy profile is an MSNE iff $\forall i \in N$

1. $u_i(s_i, \sigma^*_{-i})$ *is identical* $\forall s_i \in \delta(\sigma^*_i)$,
2. $u_i(s_i, \sigma^*_{-i}) \geqslant u_i(s'_i, \sigma^*_{-i}), \forall s_i \subseteq \delta(\sigma^*_i), s'_i \notin \delta(\sigma^*_i)$.

Observations:

- $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$
  maximizing w.r.t. a distribution $\Leftrightarrow$ whole probability mass at max

- If $(\sigma^*_i, \sigma^*_{-i})$ is an MSNE, then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma^*_{-i}) = \max_{s_i \in \delta(\sigma^*_i)} u_i(s_i, \sigma^*_{-i})$$

  the maximizer must lie in $\delta(\sigma^*_i)$ – if not, then put all probability mass on that $s'_i \notin \delta(\sigma^*_i)$ that has the maximum value of the utility – $(\sigma^*_i, \sigma^*_{-i})$ is not a MSNE

$(\Rightarrow)$ Given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \tag{1}$$

By definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \tag{2}$$

Equations (1) and (2) are equal, i.e., max is equal to positive weighted average – can happen only when all values are same: **proves condition 1**

For **condition 2**: Suppose for contradiction, there exists $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t.
$u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$

We can shift the probability mass $\sigma^*(s_i)$ to s_i', this new mixed strategy gives a strict higher utility to player i: contradicts MSNE

This completes the proof of the necessary direction.

($\Leftarrow$) Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\text{Let} \quad u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*) \qquad \textbf{condition 1}$$
$$\text{Note} \quad m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \qquad \textbf{condition 2}$$

$$
\begin{aligned}
u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*), && \textbf{by definition of } \delta(\sigma_i^*) \\
&= m_i(\sigma_{-i}^*) && \textbf{previous conclusion} \\
&= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) && \textbf{previous conclusion} \\
&= \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) && \textbf{from the observation} \\
&\geqslant u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i
\end{aligned}
$$

This proves the sufficient direction. The result yields an algorithmic way to find MSNE

Contents

Consider a NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

The total number of supports of $S_1 \times S_2 \times S_3 \cdots \times S_n$ is
$K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$

For every support profile $X_1 \times X_2 \times \cdots X_n$, where $X_i \subseteq S_i$, solve the following feasibility program

<div align="center">

Program

$$w_i = \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

$$w_i \geqslant \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

$$\sigma_j(s_j) \geqslant 0, \forall s_j \in S_j, \forall j \in N, \qquad \sum_{s_j \in X_j} \sigma_j(s_j) = 1, \forall j \in N$$

</div>

Program

$$w_i = \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

$$w_i \geqslant \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

$$\sigma_j(s_j) \geqslant 0, \forall s_j \in S_j, \forall j \in N, \qquad \sum_{s_j \in X_j} \sigma_j(s_j) = 1, \forall j \in N$$

- This is not a linear program unless $n = 2$
- For general game, there is no poly-time algorithm
- Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs] [1]

---

[1] Daskalakis, Goldberg, Papadimitriou, "The Complexity of Computing a Nash Equilibrium" [2009]

MSNE and Dominance

The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

Is there a dominated strategy in this game? Domination can be via mixed strategies too

Player 2

	L	R
T	4, 1	2, 5
M	1, 3	6, 2
B	2, 2	3, 3

Player 1

Theorem

If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta S_i$, then in every MSNE of the game, s_i is chosen with probability zero.

So, We can remove such strategies without loss of equilibrium

Contents

Definition (Finite Games)

A game is said to be **finite** when the number of players is finite, and each player has a finite set of strategies.

Theorem (Nash 1951)

Every finite game has a (mixed) Nash equilibrium.

Proof requires a few tools and a result from real analysis. Proof is separately given in the course webpage.

Some background for understanding the proof.

- A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$, $\lambda x + (1 - \lambda) y \in S$.
- A set $S \subseteq \mathbb{R}^n$ is **closed** if it contains all its **limit points** (points whose **every** neighborhood contains a point in S). Example of a set that is **not closed**: $[0,1)$ - every ball of radius $\epsilon > 0$ around 1 has a member of $[0,1)$, but 1 is not in the set $[0,1)$.
- A set $S \subseteq \mathbb{R}^n$ is **bounded** if $\exists x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$ s.t. $\forall x \in S, ||x - x_0||_2 < R$.
- A set $S \subseteq \mathbb{R}^n$ is **compact** if it is **closed** and **bounded**.

A result from real analysis (proof omitted):

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is **convex** and **compact** and $T : S \to S$, is **continuous** then T has a fixed point, i.e., $\exists\, x^* \in S$ s.t. $T(x^*) = x^*$.

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay