भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 4

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Slide preparation acknowledgments: Onkar Borade and Rounak Dalmia

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Recap

- MSNE $\rightarrow$ weakest notion of equilibrium so far

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- Existence is guaranteed for finite games

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- MSNE → weakest notion of equilibrium so far
- Existence is guaranteed for finite games
- Finding MSNE is computationally expensive

Contents

Correlated Strategy and Equilibrium

Alternative approach - entry of a **mediating** agent/device

Why do we need such an agent?

- Alternative explanation of player rationality

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- Utility for all players may get better
- Computational tractability

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Busy cross road game

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Nash solutions for the above are

- One waits and the other goes, or
- Large probability of waiting

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- **Role**:
  - randomize over the **strategy profiles** (and not individual strategies)
  - and suggest the corresponding strategies to the players
- If the strategies are **enforceable** then it is an equilibrium (**correlated**)

Correlated Strategy and Equilibrium (contd.)

Definition (Correlated Strategy)

A **correlated strategy** is a mapping $\pi : S_1 \times S_2 \times \cdots \times S_n \to [0, 1]$ s.t. $\sum_{s \in S} \pi(s) = 1$.

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A *correlated strategy* is a **correlated equilibrium** when no player *gains* by deviating from the suggested strategy while others follow the suggested strategies

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The correlated strategy π is a common knowledge

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Discussions:

- The mediator suggests the actions after running its randomization device π
- Every agent's best response is to follow it if others are also following it

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Some examples (upcoming)

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Football or Cricket Game

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Question

Are there other CEs of this game?

Contents

Computing Correlated Equilibrium

CE finding is to solve a set of linear equations

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Two set of constraints

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m^n inequalities

Computing Correlated Equilibrium (contd.)

- The inequalities together represent a **feasibility linear program** that is easier to compute than MSNE

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[1] take log of both quantities to understand this point

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- **MSNE** : total number of support profiles = $O(2^{mn})$
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- Moreover, this can also be used to optimize some objective function, e.g., maximize the sum of utilities of the players

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Comparison with the previous equilibrium notions

Theorem

For every ***MSNE*** σ^**, there exists a* ***CE*** π^*

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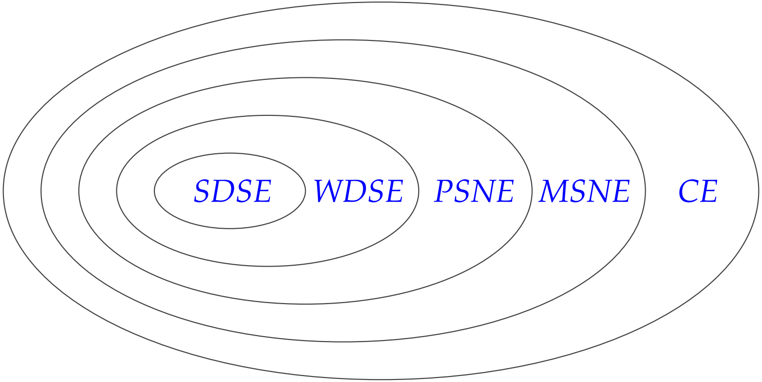
Proof Hint: Use $\pi^*(s_i, \ldots, s_n) = \prod_{i=1}^{n} \sigma_i^*(s_i)$ and the MSNE characterization theorem [**Homework**]

Venn diagram of games having equilibrium



Summary so far

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- Trusted mediator - correlated strategies - equilibrium

Richer representation of games

- More appropriate for multi-stage games, e.g. **chess**

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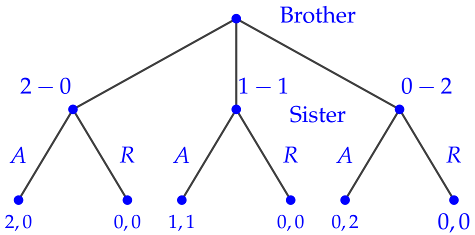
- Players interact in a sequence - the sequence of actions is the history of the game

Contents

Perfect Information Extensive Games (PIEFG)

- Brother-sister Chocolate Division
- **Disagreement** → both chocolates taken away

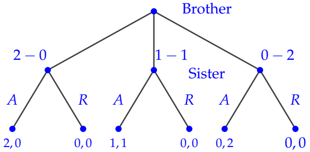


Perfect Information Extensive Form Games (PIEFG)

Formal capture

PIEFG $\langle N, A, H, X, P, (u_i)_{i \in N} \rangle$

- N: a set of players



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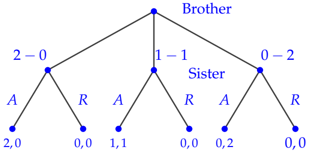

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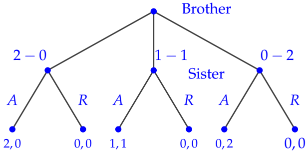

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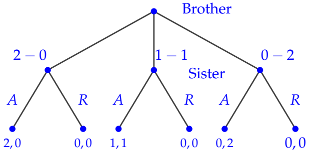

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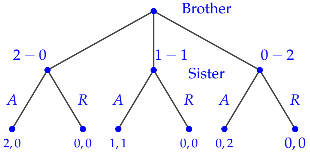

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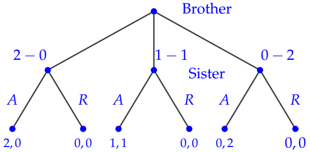

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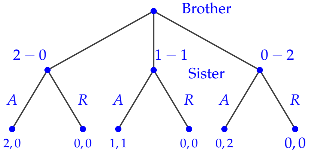

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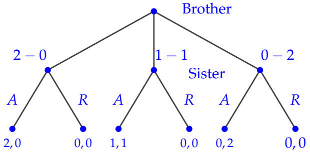

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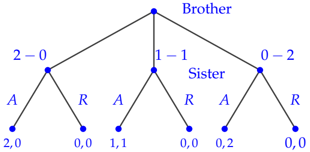

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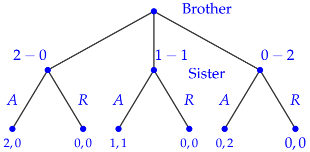
- $P : H \setminus Z \to N$: player function
- $u_i : Z \to \mathbb{R}$: utility of i



Perfect Information Extensive Form Games (PIEFG)

The **strategy** of a player in an EFG is a tuple of actions at every history where the player plays, i.e.,

$$S_i = \times_{\{h \in H : P(h) = i\}} X(h)$$

Remember:

- Strategy is a **complete contingency plan** of the player

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Remember:

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- It enumerates potential actions a player can take at every node where she can play, even though some combination of actions may never be executed together

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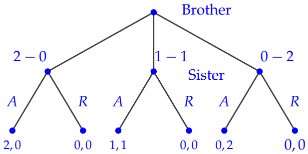

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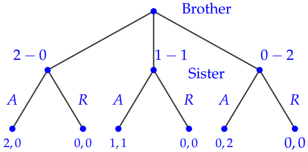

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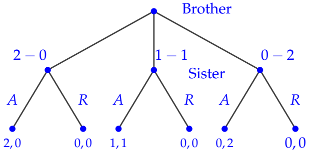

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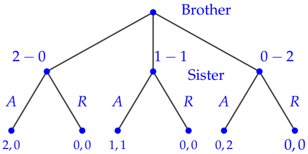

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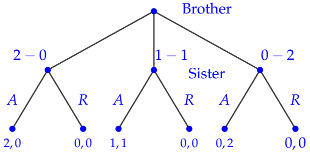

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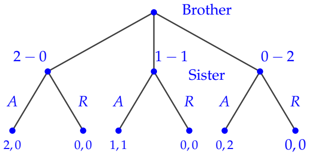

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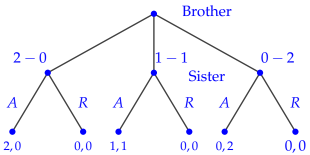

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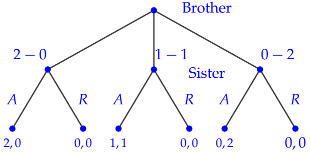

- $u_1(2-0, A) = 2, u_1(1-1, A) = 1, u_2(1-1, A) = 1, u_2(0-2, A) = 2$ [utilities are zero at the other terminal histories]

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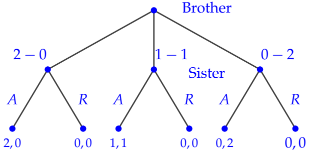

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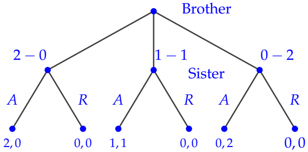

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- $S_1 = \{2-0, 1-1, 0-2\}$
- $S_2 = \{A, R\} \times \{A, R\} \times \{A, R\} = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR\}$

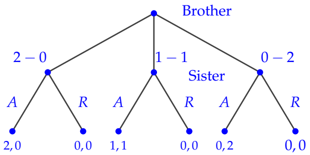
Transforming PIEFG into NFG

Once we have the S_1 and S_2, the game can be represented as an NFG

Brother \ Sister	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	$2, 0$	$2, 0$	$2, 0$	$2, 0$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
1-1	$1, 1$	$1, 1$	$0, 0$	$0, 0$	$1, 1$	$1, 1$	$0, 0$	$0, 0$
0-2	$0, 2$	$0, 0$	$0, 2$	$0, 0$	$0, 2$	$0, 0$	$0, 2$	$0, 0$

Transforming PIEFG into NFG

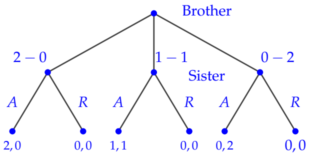
		Sister							
		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
Brother	2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
	1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
	0-2	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0



- Nash equilibrium like $(2-0, RRA)$ not quite reasonable, e.g., why R at $1-1$?

Transforming PIEFG into NFG

		Sister							
		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
Brother	2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
	1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
	0-2	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0



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- Similarly, $(2-0, RRR)$ is not a **credible threat**, i.e., if the game ever reaches the history $1-1$, Player 2's rational choice is not R

Transforming PIEFG into NFG

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2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
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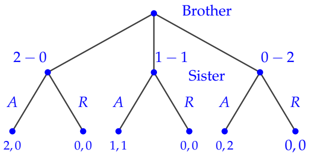

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- Hence this equilibrium concept (PSNE) is not good enough for predicting outcomes in PIEFGs

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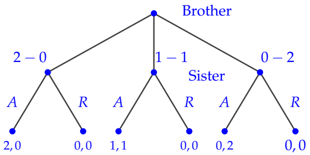

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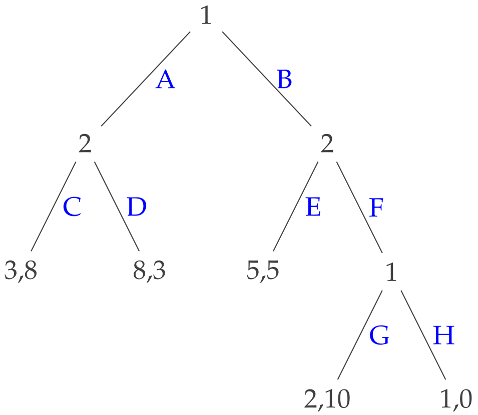
- Also the representation of a sequential game as NFG has huge redundancy – EFG is succinct

Contents

PIEFG to NFG

Equilibrium guarantees are weak for PIEFG in an NFG representation



- Strategies of Player 1 : AG, AH, BG, BH

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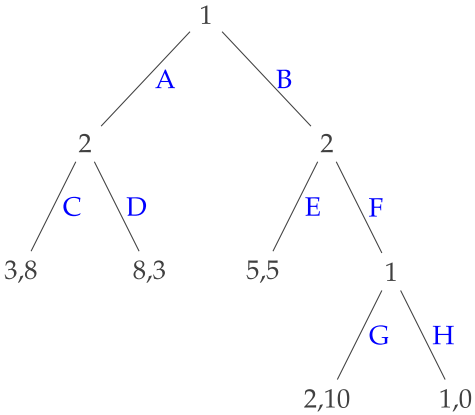

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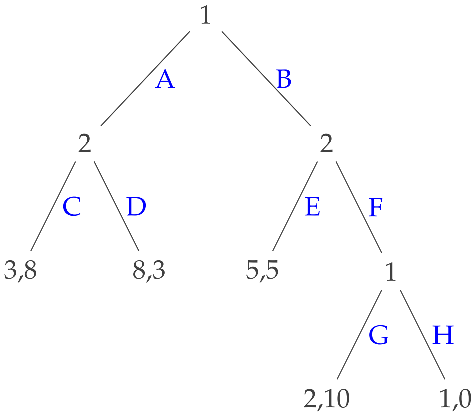

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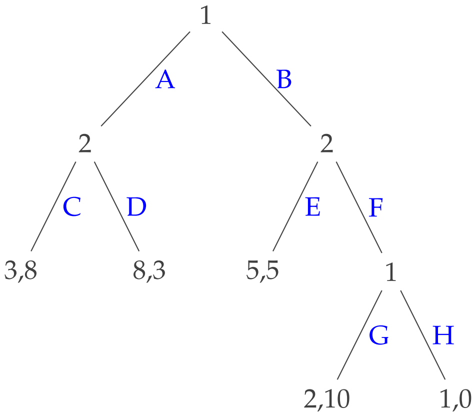

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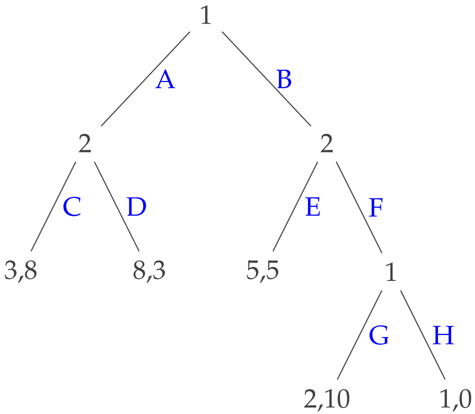

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- PSNEs?
- (AG, CF), (AH, CF), (BH, CE) – is there any non-credible threat
- Better notion of rational outcome will be that which considers a history and ensures utility maximization

Subgame and subgame perfection

Subgame: Game rooted at an intermediate vertex

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Definition (Subgame)

The subgame of a PIEFG G rooted at a history h is the *restriction* of G to the descendants of h.

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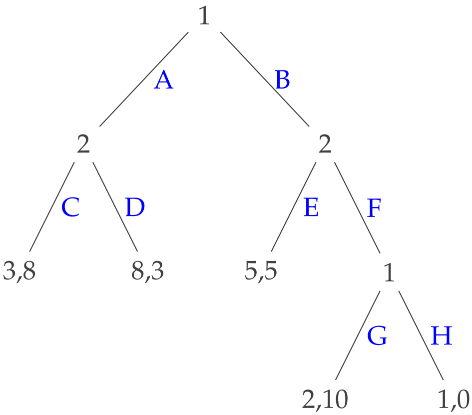
Subgame perfection: Best response at every subgame

Definition (Subgame Perfect Nash Equilibrium (SPNE))

A subgame perfect Nash Equilibrium (SPNE) of a PIEFG G is a strategy profile $s \in S$ s.t. for every subgame G' of G, the restriction of s to G' is a PSNE of G'

Example



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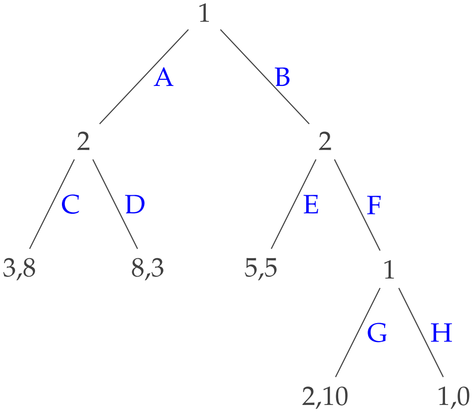

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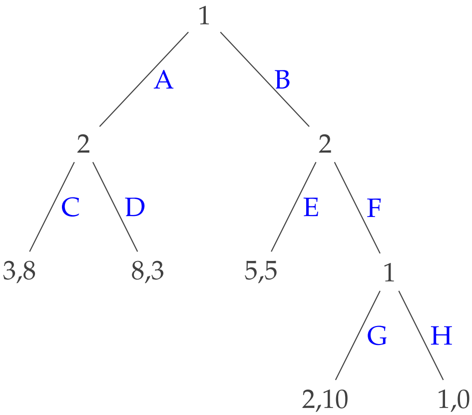

- PSNEs : (AH, CF), (BH, CE), (AG, CF)
- Are they all SPNEs?
- How to compute them?

Subgame Perfection

Algorithm 1: Backward Induction

```
Function $BACK\_IND(history\ h)$:
    if $h \in Z$ then
        return $u(h), \emptyset$
    $best\_util_{P(h)} \longleftarrow -\infty$
    foreach $a \in X(h)$ do
        $util\_at\_child_{P(h)} \longleftarrow BACK\_IND((h, a))$
        if $util\_at\_child_{P(h)} > best\_util_{P(h)}$ then
            $best\_util_{P(h)} \longleftarrow util\_at\_child_{P(h)}, best\_action_{P(h)} \longleftarrow a$
    return $best\_util_{P(h)}, best\_action_{P(h)}$
```

Contents

Limitations of SPNE

The idea of subgame perfection inherently is based on backward induction

Advantages:

- SPNE is guaranteed to exist in finite PIEFGs (requires proof)

Disdvantages and criticisms:

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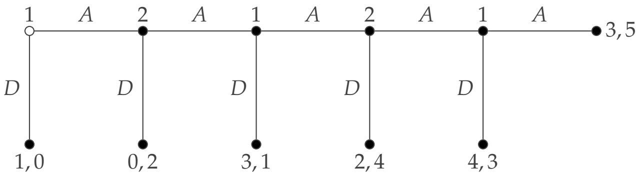
- Cognitive limit of real players may prohibit playing an SPNE

Centipede game



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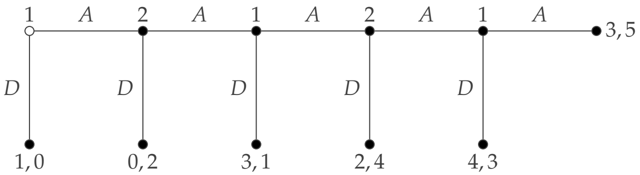

Question

What is/are the SPNE(s) of this game?

Question

What is the problem with that prediction ?

Arguments

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- Works in explaining outcomes in certain games, but there is another way to extend this idea

Arguments

- This game has been experimented with various populations
- Random participants, university students, grandmasters, etc.
- Most of the subjects (except grandmasters) continue till a few rounds (and not quit at the first round)
- **Reasons claimed:** altruism, limited computational capacity of individuals, incentive difference
- **Criticism of the defining principle of SPNE:** It talks about "what action if the game reached this history" but the equilibrium in some stage above can show that it "cannot reach that history"
- Works in explaining outcomes in certain games, but there is another way to extend this idea
- Using the idea of **belief** of the players

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