भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 5

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ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Games with Imperfect Information

The story so far

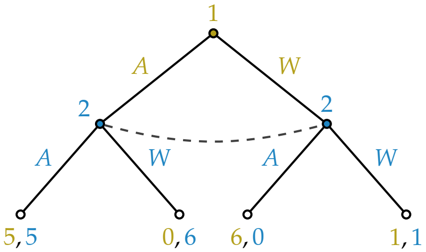
- Games discussed so far (EFGs) are of perfect information
- Every player has perfect knowledge about all the developments in the game until that round
- Limited use in certain setups:
  - several games have states that are unknown to certain agents, e.g., card games like poker, reconnaissance blind chess[a]
  - not possible to represent simultaneous move games using EFGs

[a] https://rbc.jhuapl.edu/

Games with Imperfect Information

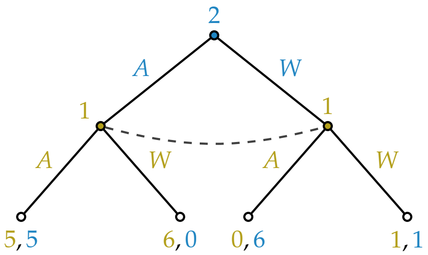
		Kingdom 2	
		Agri	War
Kingdom 1	Agri	5, 5	0, 6
	War	6, 0	1, 1

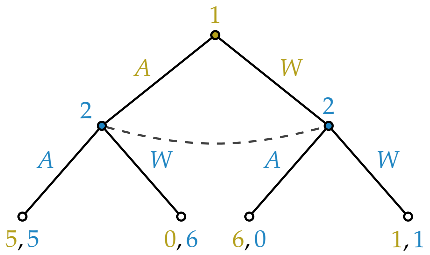
Neighboring Kingdom's Dilemma



- In IIEFG, indistinguishable nodes are connected via a dotted line.
- Kingdom 2 does not know which node/history the game is in
- These indistinguishable histories form an **information set** for player 2.
- More general representation than PIEFG since information sets can be singleton

Games with Imperfect Information





- The Neighboring Kingdom's dilemma can also be represented with the information set of player 1 being non-singleton.
- IIEFGs are not unique for a given simultaneous move game

Games with Imperfect Information

Definition (IIEFG)

An IIEFG is tuple $\langle N, A, H, X, P, (u_i)_{i\in N}, (I_i)_{i\in N}\rangle$ where $\langle N, A, H, X, P, (u_i)_{i\in N}\rangle$ is a PIEFG and for every $i \in N$, $I_i = (I_i^1, I_i^2, I_i^3, \ldots, I_i^{k(i)})$ is a partition of $\{h \in H \setminus Z : P(h) = i\}$ with the property that $X(h) = X(h')$ and $P(h) = P(h') = i$, whenever $\exists j$ s.t. $h, h' \in I_i^j$.

- I_i^js are called an **information set** of player i and I_i is the collection of information sets of i.
- At an information set, the player and her available actions are the same.
- The player is uncertain about which history in the information set is reached.

Games with Imperfect Information (contd.)

Definition (IIEFG)

An IIEFG is tuple $\langle N, A, H, X, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$ where $\langle N, A, H, X, P, (u_i)_{i \in N} \rangle$ is a PIEFG and for every $i \in N$, $I_i = (I_i^1, I_i^2, I_i^3, \dots, I_i^{k(i)})$ is a partition of $\{h \in H \setminus Z : P(h) = i\}$ with the property that $X(h) = X(h')$ and $P(h) = P(h') = i$, whenever $\exists j$ s.t. $h, h' \in I_i^j$.

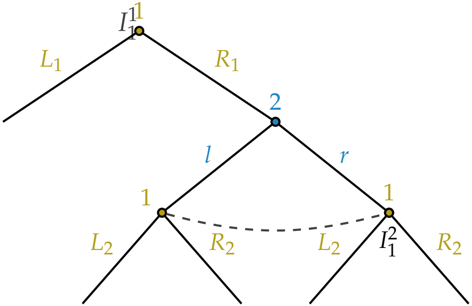
- Some differences with PIEFG
  - Since actions at an information set are identical, X (action set function) can be defined over I_i^js i.e., $X(h) = X(h') = X(I_i^j), \forall h, h' \in I_i^j$
  - Strategies can also be defined over information sets, i.e., strategy set of a player $i \in N$ is defined as the Cartesian product of actions available to i at her information sets $S_i = \times_{I' \in I_i} X(I') = \times_{j=1}^{j=k(i)} X(I_i^j)$
- With IIEFG, NFGs can be represented using EFGs, although not very succinct.
- IIEFG is a richer representation than both NFG and PIEFG.

Contents

Randomized Strategies in IIEFGs

- Strategy set of $i : S_i = \times_{j=1}^{j=k(i)} X(I_i^j)$
- In NFGs, mixed strategies randomize over pure strategies
- In EFGs, randomization can happen in different ways
  - randomize over the strategies defined at the beginning of the game
  - randomize over the action at an information set: **behavioral strategy**

Randomized Strategies in IIEFGs



- Strategies?
- Pure Strategies $(L_1 L_2)$, $(L_1 R_2)$, $(R_1 L_2)$, $(R_1 R_2)$.
- Mixed Strategy σ_1, $\sigma_1(L_1 L_2)$, $\sigma(L_1 R_2)$, $\sigma(R_1 L_2)$, $\sigma(R_1 R_2)$.
- Behavioral Strategy b_1, $b_1(I_1) \in \Delta(L_1, R_1), b_2(I_1) \in \Delta(L_1, R_1)$

Behavioral Strategy

Definition

A behavioral strategy of a player in an IIEFG is a function that maps each of her information sets to a probability distribution over the set of possible actions **at that information set**.

Mixed and Behavioral strategy

Question

What is the relation between mixed and behavioral strategies?

- In this example, MSs live in $\mathbb{R}^4$, BSs live in two $\mathbb{R}^2$ spaces
- Mixed Strategies look a 'richer' or 'larger' concept

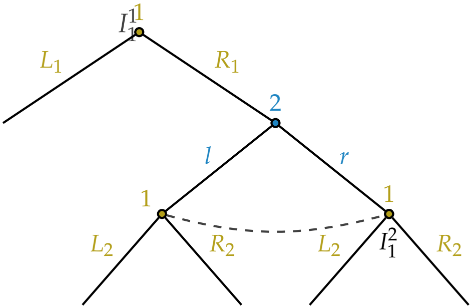
Question

Can we have an equivalence?

Equivalence in terms of the probability of reaching a vertex/history x

- Say $\rho(x;\sigma)$ is the probability of reaching a node x under mixed strategy profile σ
- Similarly, $\rho(x;b)$ is the same for behavioral strategy profile b

Example



$$\rho(x;\sigma) = \sigma_1(R_1)\sigma_2(r)$$
$$= (\sigma_1(R_1L_2) + \sigma_1(R_1R_2)) \cdot \sigma_2(r)$$
$$\rho(x;b) = b_1(I_1^1)(R_1) \cdot b_2(I_2^1)(r)$$

Players can choose different kind of strategies

$$\rho(x;\sigma_1,b_2) = (\sigma_1(R_1L_2) + \sigma_1(R_1R_2)) \cdot b_2(I_2^1)(r)$$

Equivalence Definition

Definition

A mixed strategy σ_i and a behavioural strategy b_i of a player i in an IIEFG are **equivalent** if for every mixed/behavioral strategy ξ_{-i} of the other players and every vertex x in the game tree.

$$\rho(x;\sigma_i,\xi_{-i}) = \rho(x;b_i,\xi_{-i})$$

Example (in the game above)

Equivalent strategies induce same probability of reaching a vertex.

$$\begin{aligned}
b_1(I_1^1)(L_1) &= \sigma_1(L_1L_2) + \sigma_1(L_1R_2)\\
b_1(I_1^1)(R_1) &= \sigma_1(R_1L_2) + \sigma_1(R_1R_2)\\
b_1(I_1^2)(L_2) &= \sigma_1(L_2|R_1)\\
b_1(I_1^2)(R_2) &= \sigma_1(R_2|R_1)
\end{aligned}$$

We call b_1 and σ_1 are equivalent.

More on Equivalent Strategies

The equivalence, by definition, holds at the leaf nodes too

Claim

It is enough to check the equivalence only at the leaf nodes.

Reason: Pick an arbitrary non-leaf node, the probability of reaching that node is equal to the sum of the probabilites of reaching the leaf nodes in its subtree.

More on Equivalent Strategies

This argument can be extended further

Theorem (Utility Equivalence)

If σ_i and b_i are equivalent, then for every mixed/behavioural strategy vector of the other players ξ_{-i}, the following holds,

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}) \forall j \in N.$$

Repeat the argument for any equivalent mixed and behavioral strategy profiles.

Corollary

Let σ and b are equivalent, i.e., σ_i and b_i are equivalent $\forall i \in N$, then $u_i(\sigma) = u_i(b)$.

Contents

Equivalence of strategies in IIEFGs

Question

Why behavioral strategies are desirable?

Answer

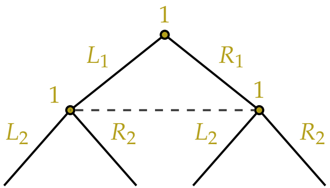
- More natural in large IIEFGs
  - players plan at every stage (information set) of the game rather than a master plan
- A smaller number of variables to deal with
  - Consider a player having 4 information sets with 2 actions each
  - needs $(2^4 - 1)$ variables to represent mixed strategies
  - needs 4 variables for behavioral strategies

Question

Can we construct one from another?
OR
Does equivalence always hold?

Equivalence of strategies in IIEFGs (Example 1)

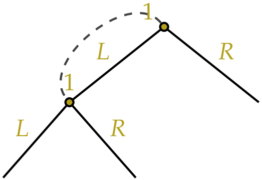
Player remembers that it made a move but forgets which move



- consider mixed strategy $\sigma_1(L_1L_2), \sigma_1(L_1R_2), \sigma_1(R_1L_2), \sigma_1(R_1R_2)$
- behavioral strategy $b_1(L_1), b_1(R_1), b_1(L_2), b_1(R_2)$.
- mixed strategy has more control over profiles, e.g., $\sigma_1(L_1R_2) = \sigma_1(R_1L_2) = 0$
- not possible in behavioral strategies
- **Mixed strategy with no equivalent behavioral strategies**

Equivalence of strategies in IIEFGs (Example 2)

Player forgets whether it made a move or not



A behavioral strategy can have a positive mass on LR, but a mixed strategy cannot.

A behavioral strategy with no equivalent mixed strategy

Answer

The equivalence does not hold if the players are forgetful

Equivalence of strategies in IIEFGs

Question

When does behavioral strategy have no equivalent mixed strategy?

Observation from a graph viewpoint

1. Let x be a non-root node
2. action at x_1 leading to x: the unique edge emanating from x_1 that is on the path from root to x
3. In example 2, there is a node that has a path from the root to itself that crosses the same information set twice
4. If the path from the root to x passes through vertices x_1 and x_1' that are in the same information set of player i, and the action leading to x at x_1 and x_1' is different, then no **pure strategy** can ever lead to x
5. Since mixed strategy is a randomization over pure strategies, every mixed strategy will put zero probability mass on x but behavioral strategy randomizes on every vertex **independently**, hence x may be reached in behavioral strategies with a positive probability

Equivalence of strategies in IIEFGs

The last observation can be stated as a lemma

Lemma

If there exists a path from the root to some vertex x that passes through the same information set at least twice, and if the action leading to x is **not** *the same at each of those vertices, then the player at the information set has a behavioral strategy that has no equivalent mixed strategy.*

This lemma helps in proving the following characterization result of equivalence.

Theorem (6.11 of MSZ)

Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy for a player iff each information set of that player intersects every path emanating from the root at most once.

Proof.

Homework. Reading exercise from MSZ. □

Contents

Behavioral Strategy equivalent to Mixed Strategy

Theorem (6.11 of MSZ)

Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy for a player iff each information set of that player intersects every path emanating from the root at most once.

To formalize (i.e., to set the conditions when the equivalence holds), we need to formalize the **forgetfulness** of a player.

- saw few examples of players' forgetfulness.
- our conditions need to ensure that none of the previous types of forgetfulness happens.

Behavioral Strategy equivalent to Mixed Strategy

Definition (Choice of same action at an information set)

Let $X = (x^0, x^1, \ldots, x^K)$ and $\hat{X} = (x_0, \hat{x}^1, \ldots, \hat{x}^L)$ be two paths in the game tree. Let I_i^j be an information set of player i that intersects these two paths only at one vertex, say x_k and x_l respectively.

These two paths **choose the same action at information set I_i^j** if

- $k < K$ and $l < L$
- actions x_k leading to x_{k+1} and $\hat{x}^l$ leading to $\hat{x}^{l+1}$ are identical, and are denoted by $a_i(x^k \to x^{k+1}) = a_i(\hat{x}^l \to \hat{x}^{l+1})$

'Leading to' may not be a relation between parent and child nodes, it can be any descendant of the former since the path is unique in a tree.

Games with Perfect Recall

Definition

Player i has perfect recall if the following conditions are satisfied

1. Every information set of player i intersects **every path from the root to a leaf at most once**.
2. Every two path that end in the same information set of player i pass through **the same information sets of i in the same order** and in every such information set **the two paths choose the same action**.

Rephrasing

For every I_i^j of player i and every pair of vertices $x, y \in I_i^j$, if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$ and $y_i^1, y_i^2, \ldots, y_i^{L'} = y$ respectively for the two paths from the root to x and y, then

1. $L = L'$,
2. $x_i^l, y_i^l \in I_i^k$ for some k,
3. $a_i(x_i^l \to x_i^{l+1}) = a_i(y_i^l \to y_i^{l+1}), \ \forall l = 1, 2, \ldots, L-1$.

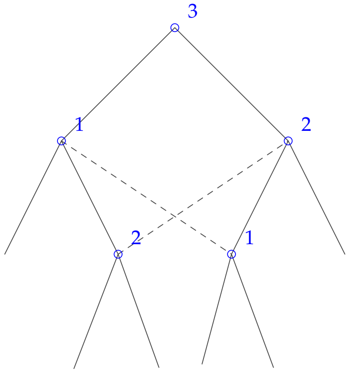
Games with Perfect Recall

Definition

A game has **perfect recall** if every player has a perfect recall.

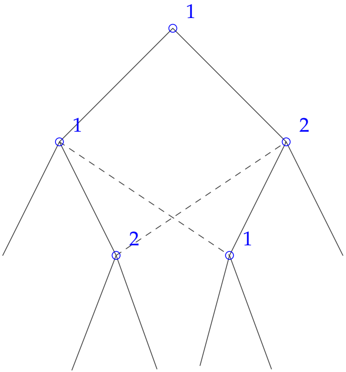
Note: Definition of perfect recall subsumes the condition where every behavioral strategy has equivalent mixed strategy

Examples



Game with Perfect Recall: This example satisfies the conditions of the definitions.

Examples



Game with Imperfect Recall: Player 1 takes two different actions at the first information set to reach two different vertices of the second information set.

Implications of Perfect Recall

Let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses actions leading to x, i.e., intersections of members of S_i with the path from root to x.

Theorem

If i is a player with perfect recall and x and x' are the two vertices in the same information set of i, then $S_i^*(x) = S_i^*(x')$.

The above conclusion comes from the same sequence of information sets and same actions. The next implication of mixed and behavioral strategies.

Implications of Perfect Recall

Theorem (Kuhn 1957)

In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of i, there exists a behavioral strategy.

- The converse is already true since the sufficient condition for that is already subsumed in the definition of perfect recall.
- Proof left as reading exercise (MSZ Theorem 6.15)
- The proof is constructive. It starts with the mixed strategy and constructs the behavioral strategies such that the probabilities of reaching a leaf are same. The arguments show that such a construction is always possible because of perfect recall.

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