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CS 6001: Game Theory and Algorithmic Mechanism Design

Week 8

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ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
- ▶ Proof of Gibbard-Satterthwaite Theorem
- ▶ Domain Restriction
- ▶ Median Voting Rule
- ▶ Median Voter Theorem: Part 1
- ▶ Median Voter Theorem: Part 2



Arrowian Social Welfare setup is too demanding

- It requires a social ordering from a preference profile
- Arrow's result says that this is impossible subject to *weak Pareto* and *independence of irrelevant alternatives* in a democratic way
- Ways out:
 - 1 consider a **social choice** setup
 - 2 put restrictions on agent preferences
- **Social choice function** (SCF)

$$f : \mathcal{P}^n \rightarrow A$$

$$A = \{a_1, a_2, \dots, a_m\}$$

$$N = \{1, 2, \dots, n\}$$

$$\mathcal{P}$$

Finite set of alternatives

Finite set of players

Set of all **linear** preference ordering



Examples

- Most representative: **voting**

$$\begin{array}{cccc}
 & & & P \\
 \hline
 a & a & c & d \\
 b & b & b & c \\
 c & c & d & b \\
 d & d & a & a
 \end{array}
 \xrightarrow{f}
 A = \{a, b, c, d\}$$

- Various voting rules exist
- **scoring rules**: each position of each agent gets a score $(s_1, s_2, \dots, s_m), s_i \geq s_{i+1}, i = 1, 2, \dots, m - 1$, the final ordering is in the decreasing order of the scores, e.g.,
 - **plurality**: $(1, 0, \dots, 0, 0)$
 - **veto**: $(1, 1, \dots, 1, 0)$
 - **Borda**: named after French mathematician Jean-Charles de Borda $(m - 1, m - 2, \dots, 1, 0)$
 - **harmonic**: $(1, 1/2, 1/3, \dots, 1/m)$
 - **k-approval**: $(\underbrace{1, 1, \dots, 1}_k, 0, 0, \dots, 0)$

Examples (contd.)



- **plurality with runoff**: also called *two round system* (TRS), first round: regular plurality and top two candidates survive, second round: another plurality **only** between the survived two candidates – used in French presidential election
- **maximin**: *maximizes the minimum lead* against other candidates: $\text{score}(a) = \min_y |\{i : aP_iy\}|$, winner is of the highest score

P			
a	a	c	d
b	b	b	c
c	c	d	b
d	d	a	a

$$\text{score}(a) = \min\{2(b), 2(c), 2(d)\} = 2$$

$$\text{score}(b) = \min\{2(a), 2(c), 3(d)\} = 2$$

$$\text{score}(c) = \min\{2(a), 2(b), 3(d)\} = 2$$

$$\text{score}(d) = \min\{2(a), 1(b), 1(c)\} = 1$$

- **Copeland**: based on Copeland score = number of wins in pairwise elections

Condorcet consistency



Definition

A voting rule is **Condorcet consistent** if it selects *the* Condorcet winner whenever one exists

- **Condorcet winner** is a candidate who defeats all other candidates in pairwise election
- Alas! it may not exist

$\begin{array}{c} P \\ \hline a \quad b \quad c \\ b \quad c \quad a \\ c \quad a \quad b \end{array}$	the voting rule can choose anything	$\begin{array}{c} P \\ \hline a \quad b \quad c \\ b \quad a \quad a \\ c \quad c \quad b \end{array}$	should choose a
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- Which of the voting rules are Condorcet consistent? plurality, Copeland, maximin?

30%	30%	40%
a	b	c
b	a	a
c	c	b

no **scoring rule** is Condorcet consistent



Desirable properties of SCF

- Recall, **social choice function**, $f : \mathcal{P}^n \rightarrow A$
- **Pareto domination**: an alternative a is **Pareto dominated** by b if $\forall i \in N, bP_i a$ (also, a is called Pareto dominated if some such b exists)

Definition (Pareto Efficiency)

An SCF f is *Pareto efficient* (PE) if $\forall P$ and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.

Definition (Unanimity)

An SCF f is *unanimous* (UN) if $\forall P$ satisfying $P_1(1) = P_2(1) = \dots = P_n(1) = a$ [$P_i(k)$ is the k -th favorite alternative of i], it holds that $f(P) = a$.

Which implies which? if the top choice of all voters is the same, say a , all other alternatives are Pareto dominated by a



Desirable properties of SCF (contd.)

Definition (Onto)

An SCF f is *onto* (ONTO) if and $\forall a \in A, \exists P^{(a)} \in \mathcal{P}^n$ s.t. $f(P^{(a)}) = a$.

UN \Rightarrow ONTO

Manipulability: an SCF f is **manipulable** if $\exists i \in N$ and a profile P such that, $f(P'_i, P_{-i}) \succ_i f(P_i, P_{-i})$, for some P'_i . Examples:

- Plurality with fixed tie-breaking

$a \succ b \succ c$

4	4	1
a	b	c
b	a	b
c	c	a

\Rightarrow

4	4	1
a	b	b
b	a	c
c	c	a

- Copeland with fixed tie-breaking

$a \succ b \succ c$

1	1	1
a	b	c
b	c	a
c	a	b

\Rightarrow

1	1	1
a	c	c
b	b	a
c	a	b

Strategyproofness and its implications



Definition (Strategyproof)

An SCF is *strategyproof* (SP) if it is not manipulable by any agent at any profile.

Implications:

- Define **dominated set** of an alternative a at a preference P_i as

$$D(a, P_i) := \{b \in A : a P_i b\}$$

- The set of alternatives *below* a in P_i

$$P_i = \begin{matrix} b \\ a \\ c \\ d \end{matrix} \Rightarrow D(a, P_i) = \{c, d\}$$



Definition (Monotonicity)

An SCF is *monotone* (MONO) if for every two profiles P and P' that satisfy $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, for all $i \in N$, it holds that $f(P') = a$.

- The relative position of c has improved from P to P' ; if c was the outcome at P , it continues to become the outcome at P'

P				P'			
-----				-----			
a	a	c	d	c	a	c	d
b	b	b	c	b	c	b	c
c	c	d	b	a	b	d	b
d	d	a	a	d	d	a	a

Theorem

An SCF f is **strategyproof** iff it is **monotone**.



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
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Strategyproofness and Monotonicity

Theorem

An SCF f is **strategyproof** iff it is **monotone**.

Proof: (SP \implies MONO)

- Consider the “if” condition of MONO
- P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i) \forall i \in N$
- Break the transition from P to P' into n stages:

$$\begin{array}{ccccc} (P_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P'_2, P_3, \dots, P_n) \\ P = P^{(0)} & & P^{(1)} & & P^{(2)} \\ \dots & \rightarrow & (P'_1, \dots, P'_k, P_{k+1}, \dots, P_n) & \rightarrow & (P'_1 \dots P'_n) \\ & & P^{(k)} & & P^{(n)} = P' \end{array}$$



Proof of SP \Leftrightarrow MONO

$$\begin{array}{lcl} (P_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P'_2, P_3, \dots, P_n) \\ P = P^{(0)} & & P^{(1)} & & P^{(2)} \\ \dots & \rightarrow & (P'_1, \dots, P'_k, P_{k+1}, \dots, P_n) & \rightarrow & (P'_1 \dots P'_n) \\ & & P^{(k)} & & P^{(n)} = P' \end{array}$$

Claim: $f(P^{(k)}) = a, \forall k = 1, \dots, n.$

- Suppose not, i.e., $\exists P^{(k-1)}, P^{(k)},$ s.t. $f(P^{(k-1)}) = a, f(P^{(k)}) = b \neq a$
- There can be one of the three cases:
 - ① $a P_k b$ and $a P'_k b \rightarrow$ voter k misreports $P'_k \rightarrow P_k$
 - ② $b P_k a$ and $b P'_k a \rightarrow$ voter k misreports $P_k \rightarrow P'_k$
 - ③ $b P_k a$ and $a P'_k b \rightarrow$ voter k misreports in both
- Contradiction to f being SP



Proof of SP \Leftrightarrow MONO (contd.)

- For (SP \Leftarrow MONO), we will prove \neg SP \implies \neg MONO
- Suppose not, i.e., f is \neg SP but MONO
- \neg SP implies that $\exists i, P_i, P'_i, P_{-i}$, s.t. $\underbrace{f(P'_i, P_{-i})}_{b \text{ (say)}} \neq \underbrace{f(P_i, P_{-i})}_{a \text{ (say)}}$
- Construct P'' s.t. $P''_{-i} = P_{-i}, P''_i(1) = b, P''_i(2) = a$
- Consider two transitions:
 - ① $(P_i, P_{-i}) \rightarrow (P''_i, P_{-i})$
 $D(a, P_i) \subseteq D(a, P''_i) \xrightarrow{\text{MONO}} f(P''_i, P_{-i}) = a$
 - ② $(P'_i, P_{-i}) \rightarrow (P''_i, P_{-i})$
 $D(b, P'_i) \subseteq D(b, P''_i) \xrightarrow{\text{MONO}} f(P''_i, P_{-i}) = b$ (contradiction)
- This concludes the proof

Equivalence of PE, UN, ONTO under SP



Lemma

If an SCF f is MONO and ONTO, then f is PE.

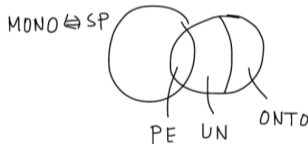


Figure: Relation between SCFs



- Suppose not, i.e. f is MONO and ONTO but not PE then $\exists a, b, P$ s.t., $b P_i a \forall i \in N$ but $f(P) = a$
- Construct P'' s.t. $P''_i(1) = b, P''_i(2) = a, \forall i \in N$
- Also $D(a, P_i) \subseteq D(a, P''_i) \forall i \in N \xrightarrow{\text{MONO}} f(P'') = a$ (contradiction)
- Hence proved

Corollary: f is SP+PE \iff f is SP+UN \iff f is SP+ONTO



Theorem (Gibbard 1973, Satterthwaite 1975)

Suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial.

The statements with f is PE (or UN) and SP are equivalent.



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
- ▶ **Proof of Gibbard-Satterthwaite Theorem**
- ▶ Domain Restriction
- ▶ Median Voting Rule
- ▶ Median Voter Theorem: Part 1
- ▶ Median Voter Theorem: Part 2



- 1 $|A| = 2$: GS theorem does not hold. Plurality with a fixed tie breaking rule is SP, ONTO, and non-dictatorial
- 2 The domain is \mathcal{P} : all permutations of the alternatives are feasible. Intuitively, every voter has many options to misreport. If the domain was limited, then GS may not hold.
- 3 **Indifference in preferences**: in general, GS theorem does not hold. In the proof, we use some specific constructions. If they are possible, then GS theorem holds.
- 4 **Cardinalization**: GS theorem will hold as long as all possible ordinal ranks are feasible in the cardinal preferences.



- For the proof, we will follow a direct approach (Sen 2001)
- First prove for $n = 2$ and then apply induction on the number of agents

Lemma

Suppose $|A| \geq 3$, $N = \{1, 2\}$, and f is ONTO and SP, then for every preference profile P , $f(P) \in \{P_1(1), P_2(1)\}$

Proof:

- If $P_1(1) = P_2(1)$, then UN implies $f(P) = P_1(1)$ (ONTO \iff UN under SP)
- Say $P_1(1) = a \neq b = P_2(1)$. For contradiction assume $f(P) = c \neq a, b$ (need at least 3 alternatives)

Proof of GS Theorem (contd.)



P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
\cdot	\cdot	\cdot	a	b	a	b	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

$$f(P_1, P_2) = c (\neq a, b)$$

- Now $f(P_1, P'_2) \in \{a, b\}$ [because all alternatives except b are Pareto dominated by a]
- But if $f(P_1, P'_2) = b$, then player 2 manipulates from P_2 to P'_2 , hence $f(P_1, P_2) = a$
- By a similar argument, $f(P'_1, P_2) = b$
- Now apply MONO
 - $P'_1, P_2 \rightarrow P'_1, P'_2$ outcome should be b
 - $P_1, P'_2 \rightarrow P'_1, P'_2$ outcome should be a (contradiction)



Proof of GS Theorem (contd.)

Lemma (Two player version of GS theorem)

Suppose $|A| \geq 3$, $N = \{1, 2\}$, and f is ONTO and SP

- Let $P : P_1(1) = a \neq b = P_2(1)$, $P' : P'(1) = c$, $P'_2(1) = d$
- If $f(P) = a$, then $f(P') = c$
- If $f(P) = b$, then $f(P') = d$

Proof: If $c = d$, unanimity proved the lemma. Hence consider $c \neq d$.

cases ↓	c	d
1	a	b
2	$\neq a, b$	b
3	$\neq a, b$	$\neq b$
4	a	$\neq a, b$
5	b	$\neq a, b$
6	b	a

- Enough to consider the case: if $f(P) = a \implies f(P') = c$
- The other case is symmetric
- These cases are exhaustive

Proof of GS Theorem (contd.)



Case 1: $c = a, d = b,$

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	a	b	a	b
\cdot	\cdot	\cdot	\cdot	b	a
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- We know (by previous lemma) $f(P') \in \{a, b\}$
- Say for contradiction $f(P') = b$

$$\begin{array}{c} P_1 \ P_2 \\ a \end{array} \xrightarrow{\text{MONO}} \begin{array}{c} \hat{P}_1 \ \hat{P}_2 \\ a \end{array}$$

$$\begin{array}{c} P'_1 \ P'_2 \\ b \end{array} \xrightarrow{\text{MONO}} \begin{array}{c} \hat{P}_1 \ \hat{P}_2 \\ b \end{array}$$



Proof of GS Theorem (contd.)

Case 2: $c \neq a, b, d = b,$

P_1	P_2	P'_1	P'_2	\hat{P}_1	P_2
a	b	c	b	c	b
\cdot	\cdot	\cdot	\cdot	a	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- We know (by previous lemma) $f(P') \in \{c, b\}$
- Say for contradiction $f(P') = b$

$$\begin{array}{c} P'_1 P'_2 \\ b \end{array} \xrightarrow{\text{MONO}}$$

$$\begin{array}{c} \hat{P}_1 P_2 \\ b \end{array}$$

(apply case 1)

agent 1 misreports $\hat{P}_1 \rightarrow P_1$ as $a \hat{P}_1 b$

Proof of GS Theorem (contd.)



Case 3: $c \neq a, b$, and $d \neq b$,

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	c	d	c	b
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- Say $f(P') = d$

$$P' \rightarrow \hat{P}$$

$$f(\hat{P}) = b \text{ (case 2)}$$

$$P \rightarrow \hat{P}$$

$$f(\hat{P}) = d \text{ (case 2)}$$

Proof of GS Theorem (contd.)



Case 4: $c = a$, and $d \neq b, a$

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	$c = a$	d	a	b
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- Say $f(P') = d$

$$P' \rightarrow \hat{P}$$

$$f(\hat{P}) = b \text{ (case 2)}$$

$$P \rightarrow \hat{P}$$

$$f(\hat{P}) = a \text{ (case 1)}$$

Proof of GS Theorem (contd.)



Case 5: $c = b$, and $d \neq b, a$

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	$c = b$	d	c	d
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- Say $f(P') = d$

$$P' \rightarrow \hat{P}$$

$$f(\hat{P}) = d \text{ (case 4)}$$

$$P \rightarrow \hat{P}$$

$$f(\hat{P}) = a \text{ (case 4)}$$



Proof of GS Theorem (contd.)

Case 6: $c = b$, and $d = a$

P_1	P_2	P'_1	P'_2	\hat{P}_1	P'_2	\tilde{P}_1	P'_2
a	b	$c = b$	$d = a$	b	a	x	a
\cdot	\cdot	\cdot	\cdot	x	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

$$f(P') = a$$

$$x \neq a, b$$

$$P' \rightarrow (\hat{P}_1 P'_2),$$

$$f(\hat{P}_1 P'_2) = a \text{ (case 1)}$$

$$P' \rightarrow (\tilde{P}_1 P'_2),$$

$$f(\tilde{P}_1 P'_2) = x \text{ (case 3)}$$

- Player 1 manipulates from $\hat{P}_1 P'_1 \rightarrow \tilde{P}_1 P'_2$, since $x \hat{P}_1 a$
- This completes the proof of $n = 2$ agent case
- **$n \geq 3$ agent case:** induction on the number of agents. See Sen (2001): "A direct proof of GS theorem", Economics Letters



- ▶ The Social Choice Setup
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- ▶ **Domain Restriction**
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- ▶ Median Voter Theorem: Part 2

GS theorem holds for unrestricted preferences



$$f : \mathcal{P}^n \rightarrow A$$

- \mathcal{P} contains all strict preferences
- One reason for a restrictive result like GS theorem is that the domain of the SCF is large
- A potential manipulator has many options to manipulate
- **Strategyproofness (an alternative definition):**

$$f(P_i, P_{-i}) \succ_i f(P'_i, P_{-i}) \text{ OR } f(P_i, P_{-i}) = f(P'_i, P_{-i}), \forall P_i, P'_i \in \mathcal{P}, \forall i \in N, \forall P_{-i} \in \mathcal{P}^{n-1}$$

- If we reduce the set of feasible preferences from \mathcal{P} to $\mathcal{S} \subset \mathcal{P}$
 - the SCF f strategyproof on \mathcal{P} continues to be strategyproof over \mathcal{S}
 - but there can potentially be more f 's that can be strategyproof on the **restricted domain**



- 1 Single peaked preferences
- 2 Divisible goods allocation
- 3 Quasi-linear preferences

Each of these domains have interesting non-dictatorial SCFs that are strategyproof

Single peaked preferences



- Temperature of a room
- For every agent, most comfortable temperature t_i^*
- Anything above or below are monotonically less preferred

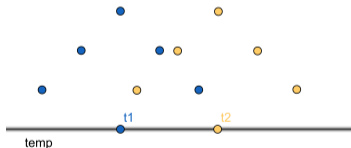


Figure: Single peaked temperature preference

Single peaked preferences

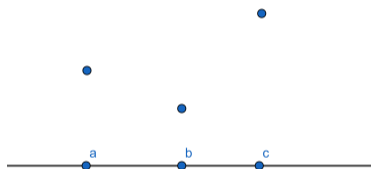


- One **common order** over the alternatives
- Agent preferences are single peaked w.r.t. that common order
- Other examples:
 - 1 Facility location: School/Hospital/Post office
 - 2 Political ideology: Left, Center, Right
- The common ordering of the alternatives is denoted via $<$ [as in real numbers]
- Any relation over the alternatives that is transitive and antisymmetric. In this course, we will assume:
 - 1 alternatives live on a real line
 - 2 consider only one-dimensional single-peakedness



Single peaked preferences

How is it a domain restriction?



Consider $a < b < c$, all possible orderings:

a	b	b	c	a	c
b	a	c	b	c	a
c	c	a	a	b	b

Definition (Single peaked preferences)

A preference ordering P_i (linear over A) of agent i is single-peaked w.r.t. the common order $<$ of the alternatives if

- 1 $\forall b, c \in A$ with $b < c \leq P_i(1)$, cP_ib
- 2 $\forall b, c \in A$ with $P_i(1) \leq b < c$, bP_ic

Single peaked preferences



- Let \mathcal{S} be the set of single peaked preferences. The SCF: $f : \mathcal{S}^n \rightarrow A$

Question

How does it circumvent GS theorem?

Answer

Each player's preference has a peak. Suppose, f picks the leftmost peak. For the agent having the leftmost peak, no reason to misreport. For any other agent, the only way she can change the outcome is by reporting her peak to be left of the leftmost – but that is strictly worse than the current outcome.

Repeat this argument for any fixed k^{th} peak from left. Even the rightmost peak choosing SCF is also strategyproof, so is the median ($k = \lfloor \frac{n}{2} \rfloor$)



- ▶ The Social Choice Setup
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- ▶ Domain Restriction
- ▶ **Median Voting Rule**
- ▶ Median Voter Theorem: Part 1
- ▶ Median Voter Theorem: Part 2



Definition

An SCF $f : \mathcal{S}^n \rightarrow A$ is a median voter SCF if there exists $B = \{y_1, y_2, \dots, y_{n-1}\}$ s.t. $f(P) = \text{median}(B, \text{peaks}(P))$ for all preference profiles $P \in \mathcal{S}$

- Here, the median is w.r.t. the common order $<$
- The points in B are called the peaks of **phantom voters**
- **Note:** B is fixed for f and does not change with P
- **Why phantom voters?**
- $f^{\text{leftmost}} \equiv (B_{\text{left}}, \text{peaks}(P)); B_{\text{left}} = \{y_L, \dots, y_L\}$, i.e., if all phantom peaks are on the left, it corresponds to leftmost peak SCF
- Similarly, $f^{\text{rightmost}}(\cdot)$ can be found in a similar way
- Phantom voters give a complete spectrum of the median voter SCFs



Theorem (Moulin 1980)

Every median voter SCF is strategyproof.

Proof Sketch:

- if $f(P) = a$ and a player has a peak $P_i(1)$ to the left of a , it has no benefit by misreporting the peak to be on the right of a , which is the only way of changing the outcome of f
- similar for $P_i(1)$ on the right of a

Note: mean does not have this property



Claim

Let p_{min} and p_{max} be the leftmost and rightmost peaks of P according to $<$, then f is PE iff $f(P) \in [p_{min}, p_{max}]$

Proof: (\implies) Suppose $f(P) \notin [p_{min}, p_{max}]$, WLOG, $f(P) < p_{min}$. Then every agent prefers p_{min} over $f(P)$, i.e., $f(P)$ is Pareto dominated. Contradiction

(\impliedby) If $f(P) \in [p_{min}, p_{max}]$, then the condition $bP_i f(P), \forall i \in N$ never occurs – there does not exist an alternative b that Pareto dominates $f(P)$. Hence $f(P)$ is PE.

Median voter SCF and Monotonicity



Definition (Monotonicity)

An SCF is *monotone* (MONO) if for every two profiles P and P' that satisfy $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, for all $i \in N$, it holds that $f(P') = a$.

- The relative position of c has improved from P to P' ; if c was the outcome at P , it continues to become the outcome at P'

P				P'			
a	a	c	d	c	a	c	d
b	b	b	c	b	b	b	c
c	c	d	b	a	c	d	b
d	d	a	a	d	d	a	a

Median voter SCF and Monotonicity



The results are similar to unrestricted preferences in this restricted domain of single peaked preferences, but the proofs differ since we cannot construct preferences as freely as before.

Theorem

f is SP $\implies f$ is MONO

This proof is similar to the previous one. To prove the reverse implication one needs to argue why the construction is valid in the single peaked domain. (or provide counterexample)



Equivalence of ONTO, UN, and PE

Theorem

Let $f : S^n \rightarrow A$ is a SP SCF. Then, f is ONTO $\iff f$ is UN $\iff f$ is PE

Proof:

- We know $PE \implies UN \implies ONTO$
- Need to show: ONTO implies PE when f is SP
- Suppose, for contradiction, f is SP and ONTO, but not PE
- Then $\exists a, b \in A$ s.t. $a P_i b \forall i \in N$ but $f(P) = b$
- Since preferences are single peaked, \exists another alternative $c \in A$, which is a neighbour of b s.t. $c P_i b \forall i \in N$ (c can be a itself)

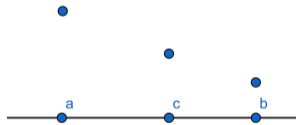


Figure: Arrangement of a, b, c



- ONTO $\implies \exists P'$ s.t. $f(P') = c$
- Construct P'' s.t. $P''_i(1) = c, P''(2) = b, \forall i \in N$
- $P \rightarrow P'', \text{ MONO} \implies f(P'') = b$
- $P' \rightarrow P'', \text{ MONO} \implies f(P'') = c$
- Contradiction

We are interested in non-dictatorial SCFs, hence a necessary property is anonymity



Anonymity

- **Anonymity:** outcome insensitive to agent identities
- Permutation of agents $\sigma : N \rightarrow N$
- We apply a permutation σ to a profile P to construct another profile as: the preference of i goes to agent $\sigma(i)$ in the new profile
- Denote this new profile as P^σ
- **Example:** $N = \{1, 2, 3\}, \sigma : \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$

P_1	P_2	P_3	P_1^σ	P_2^σ	P_3^σ
a	b	b	b	a	b
b	a	c	c	b	a
c	c	a	a	c	c

Definition

An SCF $f : S^n \rightarrow A$ is **anonymous** (ANON) if for every profile P and for every permutation of the agents $\sigma, f(P^\sigma) = f(P)$



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Median Voter Theorem



Seen the equivalence of SP, ONTO, ANON and median voting rule in single peaked domain

Theorem

Let $f : S^n \rightarrow A$ is a SP SCF. Then, f is ONTO $\iff f$ is UN $\iff f$ is PE

Theorem (Moulin 1980)

A **strategyproof** SCF f is ONTO and **anonymous** iff it is a median voter SCF.

Proof: (\iff)

- Median voter SCF is SP (previous theorem)
- It is **anonymous**: if we permute the agents with peaks unchanged, the outcome does not change
- It is ONTO, pick any arbitrary alternative a , put peaks of all players at a : the outcome will be a irrespective of the positions of the phantom peaks (since there are $(n - 1)$ phantom peaks and n agent peaks)

Proof (contd.)



\implies Given, $f : \mathcal{S}^n \rightarrow A$ is SP, ANON, and ONTO.

- define, P_i^0 : agent i 's preference with peak at leftmost w.r.t. $<$
- P_i^1 : agent i 's preference with peak at rightmost w.r.t. $<$

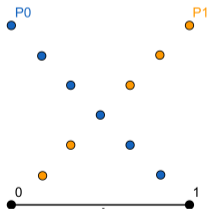


Figure: Two preferences



The proof is constructive, we will construct the median voting rule (which needs the phantom peaks to be defined) s.t. the outcome of an arbitrary f matches the outcome of the median SCF

- First construct phantom peaks

$$y_j = f(\underbrace{P_1^0, P_2^0, \dots, P_{n-j}^0}_{n-j \text{ peaks leftmost}}, \underbrace{P_{n-j+1}^1, \dots, P_n^1}_{j \text{ peaks rightmost}}), \quad j = 1, \dots, n-1$$

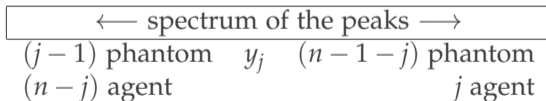
Which agents have which peaks does not matter because of anonymity

- **Claim:** $y_j \leq y_{j+1}$, $j = 1, \dots, n-2$, i.e., peaks are non-decreasing
- **Proof:** $y_{j+1} = f(P_1^0, P_2^0, \dots, P_{n-j-1}^0, P_{n-j}^1, P_{n-j+1}^1, \dots, P_n^1)$. Due to SP, $y_j \geq y_{j+1}$, or they are same, but P_{n-j}^0 is single peaked with peak at 0, hence $y_j \leq y_{j+1}$



Proof (contd.)

- Consider an arbitrary profile, $P = (P_1, P_2, \dots, P_n)$, $P_i(1) = p_i$ (the peaks)
- **Claim:** Suppose f satisfies SP, ONTO, ANON, then $f(P) = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$
- WLOG, can assume $p_1 \leq p_2 \leq \dots \leq p_n$ due to ANON
- **Case 1:** a is a phantom peak, say $a = y_j$ for some $j \in \{1, 2, \dots, n-1\}$
- This is a median of $2n-1$ points of which $(j-1)$ phantom peaks lie on the left (see the claim before), the rest $(n-j)$ points are agent peaks



- Hence, $p_1 \leq \dots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \dots \leq p_n$



- Use a similar transformation as we used earlier

$$f(P_1^0, P_2^0, \dots, P_{n-j}^0, P_{n-j+1}^1, \dots, P_n^1) = y_j \text{ (definition)}$$

$$f(P_1, P_2^0, \dots, P_{n-j}^0, P_{n-j+1}^1, \dots, P_n^1) = b \text{ (say)}$$

$$\text{By SP, } y_j P_1^0 b \implies y_j \leq b$$

$$\text{Again by SP, } b P_1 y_j, \text{ but } p_1 \leq y_j \xrightarrow{\text{single peaked}} b \leq y_j$$

$$\text{Hence, } b = y_j$$

- repeat this argument for the first $(n - j)$ agents to get

$$f(P_1, P_2, \dots, P_{n-j}, P_{n-j+1}^1, \dots, P_n^1) = y_j$$



Proof (contd.)

- We have

$$f(P_1, P_2, \dots, P_{n-j}, P_{n-j+1}^1, \dots, P_n^1) = y_j$$

- Consider

$$f(P_1, P_2, \dots, P_{n-j}, P_{n-j+1}^1, \dots, P_n) = b \text{ (say)}$$

- Apply very similar argument

$$\left. \begin{array}{l} y_j P_n^1 b \implies b \leq y_j \\ b P_n y_j \text{ and } y_j \leq p_n \implies y_j \leq b \end{array} \right\} b = y_j$$

- Hence,

$$f(P - 1, \dots, P_n) = y_j$$



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Median Voter Theorem: Proof

- The claim we are proving
- **Claim:** Suppose f satisfies SP, ONTO, ANON, then $f(P) = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$
- WLOG, can assume $p_1 \leq p_2 \leq \dots \leq p_n$ due to ANON
- **Case 1:** a is a phantom peak: proved
- **Case 2:** a is an agent peak
- We will prove this for 2 players, the general case repeats this argument
- **Claim:** $N=\{1,2\}$, let P and P' be such that $P_i(1) = P'_i(1), \forall i \in N$. Then $f(P) = f(P')$
- **Proof:** Let $a = P_1(1) = P'_1(1)$, and $P_2(1) = P'_2(1) = b$. $f(P) = x$ and $f(P'_1, P_2) = y$
- Since f is SP, $x P_1 y$ and $y P'_1 x$
- Since peaks of P_1 and P'_1 are the same, if x, y are on the same side of the peak, they must be the same, as the domain is single peaked
- The only other possibility is that x and y fall on different sides of the peak: **we show that this is not possible.**



Proof (contd.)

- WLOG $x < a < y$ and $a < b$
- f is SP+ONTO \iff f is SP+PE
- PE requires $f(P) \in [a, b]$, but $f(P) = x < a \rightarrow \leftarrow$
- Repeat this argument for $(P'_1, P_2) \rightarrow (P'_1, P'_2) \square$

Profile: $(P_1, P_2) = P, P_1(1) = a, P_2(1) = b, y_1$ is the phantom peak, and by assumption, $\text{median}(a, b, y_1)$ is an agent peak

- WLOG assume that the median is a
- Assume for contradiction $f(P) = c \neq a$
- By PE, c must be within a and b
- We have two cases to consider: $b < a < y_1$ and $y_1 < a < b$



Case 2.1: $b < a < y_1$, by PE $c < a$

- Construct P'_1 s.t. $P'_1(1) = a = P_1(1)$ and $y P'_1 c$ (possible since they are on different sides of a)
- By the earlier claim, $f(P) = c \implies f(P'_1, P_2) = c$
- Now consider the profile (P'_1, P_2) (P'_1 has its peak at the rightmost point)
- $P_2(1) = b < y \leq P'_1(1)$, hence the median of $\{b, y_1, P'_1(1)\}$ is y_1 (which is a phantom peak, hence case 1 applies)
- We get $f(P'_1, P_2) = y_1$
- But $y P'_1 c$ (by construction) and $f(P'_1, P_2) = c$
- Agent 1 manipulates $P'_1 \rightarrow P_1^1$, contradiction to f being SP



Case 2.2: $y_1 < a < b$, by PE $a < c$

- Construct P'_1 s.t. $P'_1(1) = a = P_1(1)$ and $y P'_1 c$
- $f(P'_1, P_2) = c$ (by claim)
- Consider (P_1^0, P_2) , $P_1^0(1) \leq y_1 < b \implies f(P_1^0, P_2) = y_1$ but $y_1 P'_1 c$, hence manipulable by agent 1
- **This completes the proof for two agents (case 2)**
- For the generalization to n players, see Moulin (1980) "On strategyproofness and single-peakedness"



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