भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 10

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Slide preparation acknowledgments: Onkar Borade

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

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The Vickrey-Clarke-Groves Mechanism (VCG)

- The most popular mechanism in the Groves class
- Also known as the pivotal mechanism (V'61, C'71, G'73)
- Given by a unique $h_i(\theta_{-i})$ function in the Groves class

$$h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

$$p_i^{VCG}(\theta_i, \theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j)$$

- Interpretation of the **payment**: **sum value of others (in absence of i − in presence of i)**
- Interpretation of the **utility** under VCG mechanism:

$$v_i(f^{AE}(\theta_i, \theta_{-i}), \theta_i) - p_i^{VCG}(\theta_i, \theta_{-i}) = \sum_{j \in N} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

$$= \text{marginal contribution of } i \text{ in the social welfare}$$

An Observation on VCG Mechanism

Utility under VCG mechanism:

$$v_i(f^{AE}(\theta_i, \theta_{-i}), \theta_i) - p_i^{VCG}(\theta_i, \theta_{-i}) = \sum_{j \in N} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

- **Note:** utility is non-negative, i.e., VCG is **individually rational** (for **public goods** or **object allocation**)
- Can be generalized using properties **choice set monotonicity** and **no negative externality** – later
- Intuition:
  - **choice set monotonicity** says that with more agents, set of alternatives never reduces
  - **no negative externality** says if an agent is removed, that agent does not have a negative value towards the AE allocation without that agent

Examples

1. **Single Object Allocation**
   Type = value of the object if allocated, the agent get this value and zero otherwise

$$p_i^{VCG}(\theta_i, \theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j) \quad (1)$$

   Efficient allocation would give the object to the allocation whose reported type is highest
   - Consider 4 players , types $\{10, 8, 9, 5\} \implies$ item is given to player 1, and payments are: $\{9, 0, 0, 0\}$
   - This is **second price auction**

Examples

2. What is pivotal in the VCG payment?
   3 players having the following valuations :

	Football	Library	Museum
A	0	70	50
B	95	10	50
C	10	50	50

- VCG allocation: M (maximizes social welfare)
- Payments
  A pays $= 105 - 100 = 5$
  B pays $= 120 - 100 = 20$
  C pays $= 100 - 100 = 0 \leftarrow$ **non-pivotal agent**
- The agent whose presence *changes the outcome* is charged money
  They are the **pivotal** players

Examples

3. Combinatorial Allocation: sale of multiple objects
   3 players having the following valuations (value is the type itself $v_i(a, \theta_i) = \theta_i(a)$)

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1\&2\}$
θ_1	0	8	6	12
θ_2	0	9	4	14

— Efficient allocation : $\{1\} \to 2$ & $\{2\} \to 1$: **Call this a^***
— $p_1^{VCG}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq 1} \theta_j(a) - \sum_{j \neq 1} \theta_j(a^*) = 14 - 9 = 5$, payoff $= 6 - 5 = 1$
— $p_2^{VCG}(\theta_1, \theta_2) = 12 - 6 = 6$, payoff $= 9 - 6 = 3$

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VCG Mechanism in Combinatorial Allocations

- $M = \{1, \ldots, m\}$: set of objects
- $\Omega = 2^{\{1,\ldots,m\}}$: set of bundles
- $\theta_i : \Omega \to \mathbb{R}$: type/value of agent i
- We assume $\theta_i(s) \geqslant 0, \forall s \in \Omega$, objects are **goods**
- An allocation in this case is a partition of the objects, i.e.,

$$a = \{a_0, a_1, a_2, \ldots, a_n\}, \; a_i \in \Omega, \; a_i \cap a_j = \emptyset, \; \forall i \neq j$$

$$a_0 : \text{ set of unallocated objects}, \cup_{i=0}^{n} a_i = M$$

  Let A be the set of all such allocations

- Also assume **selfish valuations**, i.e., $\theta_i(a) = \theta_i(a_i)$, agent i's valuation does *not* depend on the allocations of others

VCG Mechanism in Combinatorial Allocations

Claim

In the allocation of goods, the VCG payment for an agent, that gets no object in this efficient allocation, is zero

Proof sketch:

$$a^* \in \arg\max_{a \in A} \sum_{j \in N} \theta_j(a), a_i^* = \emptyset$$

$$a_{-i}^* \in \arg\max_{a \in A} \sum_{j \in N \setminus \{i\}} \theta_j(a)$$

- Note, $p_i^{VCG}(\theta) \geqslant 0$, also $p_i^{VCG}(\theta) = \sum_{j \neq i} \theta_j(a_{-i}^*) - \sum_{j \neq i} \theta_j(a^*)$
- **Note:** $\theta_i(a_{-i}^*) = 0$, and $\theta_i(a^*) = \theta_i(a_i^*) = 0$
- Add the first to the first term and subtract the second from the second term above
- $p_i^{VCG}(\theta) = \sum_{j \in N} \theta_j(a_{-i}^*) - \sum_{j \in N} \theta_j(a^*) \leqslant 0 \implies p_i^{VCG}(\theta) = 0$

VCG Mechanism in Combinatorial Allocations

Definition (Individual Rationality)

A mechanism (f, p) is *individually rational* if $v_i(f(\theta), \theta_i) - p_i(\theta) \geqslant 0, \forall \theta \in \Theta, \forall i \in N$

Claim

In the allocation of goods, the VCG mechanism is individually rational

$$
\begin{aligned}
\text{utility of player } i = \theta_i(a^*) - p_i^{VCG}(\theta) &= \theta_i(a^*) - \Big(\sum_{j \neq i} \theta_j(a^*_{-i}) - \sum_{j \neq i} \theta_j(a^*)\Big) \\
&= \sum_{j \in N} \theta_j(a^*) - \sum_{j \neq i} \theta_j(a^*_{-i}) {\color{red}- \theta_i(a^*_{-i}) + \theta_i(a^*_{-i})} \\
&= \underbrace{\sum_{j \in N} \theta_j(a^*) - \sum_{j \in N} \theta_j(a^*_{-i})}_{\geqslant 0, \text{ by definition of } a^*} + \underbrace{\theta_i(a^*_{-i})}_{\geqslant 0} \geqslant 0
\end{aligned}
$$

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Application domain: Internet advertising

The reason for success of Internet advertising:

1. **User data**
   — Advertisers can gather a lot of data from the user to design targeted products
2. **Measurable Actions**
   — Can classify buyers into categories and measure the interest and take appropriate actions
3. **Low Latency**
   — Real-time bidding, automated bidding, decisions on the fly possible

Types of advertisements on the Internet

1. **Sponsored Search Ads**
   — Advertisers bid on the keywords entered by the user during search
2. **Contextual Ads**
   — depending on the context of the page, email or post message
3. **Display Ads**
   — Traditional modes of advertising, e.g., banner ads in newspapers

- Ads are complex - modern internet advertising is handled via **ad exchanges**
- Small businesses can customize these ads via exchanges

Position Auctions

- **Position Auctions:** auctions to sell multiple ad positions on a page
- Let $N = \{1, 2, \ldots, n\}$: set of advertisers
- Let $M = \{1, 2, \ldots, m\}$: set of slots, assume $m \geqslant n$, i.e., every ad is shown; 1: best position, m: worst position
- **Evolution of position auctions:**
  - Early positions auctions ordered the ads via **bid-per-impression**
    - just for showing the ad, e.g., newspaper ads
    - all risk on the advertiser
  - Bids on clicks **pay-per-click model**
    - risk is shared by the publisher
    - ranked by **pay-per-click**
    - If shown ads are not clicked, the publisher earns nothing
  - Today's approach: Rank advertisers based on the product of the **probability of a click** and the **bid value**
    - Probability of click is called **click through rate (CTR)**
    - rank by expected revenue

Advertiser Value

- **Assumption 1:** Clicks generate value to the advertisers
- **Assumption 2:** All clicks are valued equally, no matter what position the ad is displayed - the position only affects the chance of getting the click
- These assumptions help decouple the value effect and position effect
- Agent i's expected value when her ad is shown at position $j \in M$ is given by

$$v_{ij} = CTR_{ij} \cdot v_i$$

  $CTR_{ij} \in [0, 1]$: **click through rate**, i.e., probability of getting a click on i's ad at j^{th} position, v_i: **value of a click**
- Further assumption: $CTR_{ij} = \rho_i \cdot p_j$, where ρ_i: quality component, and p_j: position component
- Hence, agent i's expected value when her ad is shown at position $j \in M$

$$v_{ij} = \underbrace{p_j}_{\text{position effect}} \cdot \underbrace{(\rho_i v_i)}_{\text{agent effect}}$$

Advertiser Value

$$v_{ij} = \underbrace{p_j}_{\text{position effect}} \cdot \underbrace{(\rho_i v_i)}_{\text{agent effect}}$$

- Position effect is assumed to be decreasing with position

$$p_1 = 1, p_j > p_{j+1}, \forall j = 1, \ldots, m-1$$

- v_i is the only private information of the advertiser
- p_j and ρ_i are measurable
- Search engines estimate the ρ_i: say $\hat{\rho}_i$
- Bidders bid b_i
- Reported agent effect component is $\hat{\rho}_i \cdot b_i$

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Allocation of Slots in Position Auctions

- Suppose the allocation of the slots is given by $a = (a_1, a_2, \ldots, a_n)$ is the allocation, where a_i is the slot allocated to i
- Then the value of agent i:
$$v_i(a, \theta_i) = p_{a_i} \cdot (\hat{\rho}_i \cdot \theta_i)$$
- **Efficient** allocation: $a^* \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$
- **Observe:** an allocation a is efficient iff it is a **"rank-by-expected revenue"** $(\hat{\rho}_i \cdot \theta_i)$ mechanism
- **Why?** because it is a moment maximization problem: sum is maximized when the maximum weight is put on the maximum value
- The slot allocation problem is a **sorting** problem, hence computationally tractable
- Allocation decision is done, need payments to make it DSIC
- Natural candidate: **VCG**
- **Note:** actual implementation in practice might be different, here we discuss only the principle of its computation

Allocation of Slots in Position Auctions

- VCG in the context of position auctions
- Given bids $(b_1, \cdots, b_n)$ (Note: $\hat{\theta}_i$: reported type and b_i are the same)
- WLOG, assume the order to be such that $\hat{\rho}_1 b_1 \geqslant \hat{\rho}_2 b_2 \geqslant \cdots \geqslant \hat{\rho}_n b_n$
  - allocation a^* is s.t., $a_i^* = i$
  - define $a_{-i}^* \in \arg\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$: in this allocation, the slots allocated to the agents after i, i.e., from $i+1$ to n get one slot better/above than a^*
- Hence,

$$p_i^{VCG} = \sum_{j \neq i} v_j(a_{-i}^*, \theta_j) - \sum_{j \neq i} v_j(a^*, \theta_j) = \sum_{j=i}^{n-1} p_j(\hat{\rho}_{j+1} b_{j+1}) - \sum_{j=i}^{n-1} p_{j+1}(\hat{\rho}_{j+1} b_{j+1})$$

$$= \sum_{j=i}^{n-1} (p_j - p_{j+1})(\hat{\rho}_{j+1} b_{j+1}), \ \forall i = 1, \cdots, n-1, \text{ and}$$

$$p_n^{VCG}(b) = 0$$

- This is the total expected payment, to convert this to the pay-per-click: $\frac{1}{p_i \hat{\rho}_i} p_i^{VCG}(b)$

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Pros of VCG Mechanism

1. DSIC: hence very low cognitive load on the bidders
2. No subsidy (and therefore, deficits) under certain conditions if items are goods
3. Never charges an agent who gets no items
4. Individually rational to participate: nobody loses money

Cons of VCG Mechanism

1. Privacy and transparency
   - it reveals true valuations/types. Two competing companies would not like to make private information public
   - a malicious auctioneer may introduce fake bidders to extract more payment from the bidders
2. Susceptibility to Collusion: consider a public good decision of A or B

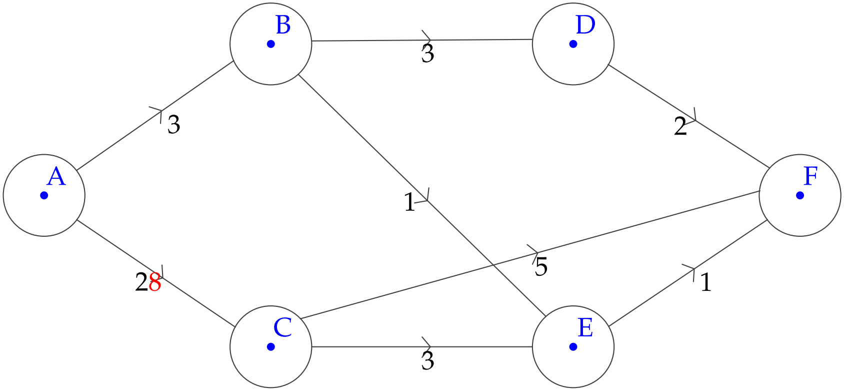
	A	B	Payment
1	200	0	150
2	100	0	50
3	0	250	0

	A	B	Payment
1	250	0	100
2	150	0	0
3	0	250	0

   - If 1 and 2 collude and bid higher, both of them reduce their payments $\implies$ utility increases

Cons of VCG Mechanism

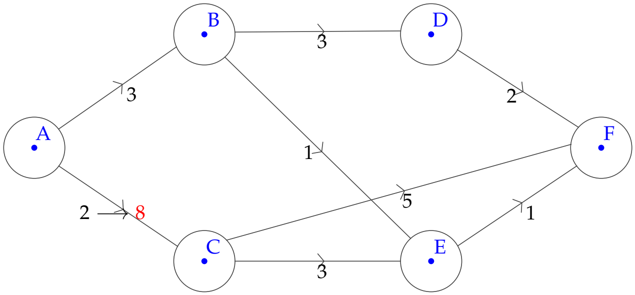
3. Not frugal: payment could be very large: VCG is guaranteed to be no deficit but can charge payments much larger than the cost



Example: Item delivery network (e.g, Amazon), source: A, destination: F

Cons of VCG Mechanism

- This is a cost setup, hence the values can be considered to be negative
- Each edge is a player
- Efficient allocation: $A \to B \to E \to F$
- $p_{AB} = (-2 - 3 - 1) - (-1 - 1) = -4$
- $p_{AB} = (-8 - 3 - 1) - (-1 - 1) = -10$



Cons of VCG Mechanism

4. Revenue monotonicity violation: revenue should weakly increase with the number of players

	F	M	Payment
1	0	90	$0 \rightarrow 0$
2	100	0	$90 \rightarrow 0$
3	100	0	0

Nobody's pivotal

5. Not budget balanced: this is a no-deficit mechanism but it almost always keeps surplus, which can be large
   - This money cannot be redistributed among the same players, since that will change their payoffs and the resulting mechanism would not remain DSIC
   - If the players are partitioned into two groups and the surplus of one group is redistributed over the other group, then it is budget balanced, but the overall efficiency is compromised

Cons of VCG and Concluding Remark

- This surplus has to be taken away or destroyed: **money burning**
- How much to burn and what efficiency we compromise?
- **Nath and Sandholm (2019): Efficiency and budget balance in general quasi-linear domains, Games and Econ Behavior**
- **But**

These are certain limitations that are good to know for effective use of VCG, however, it is the most widely used mechanism in the literature

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