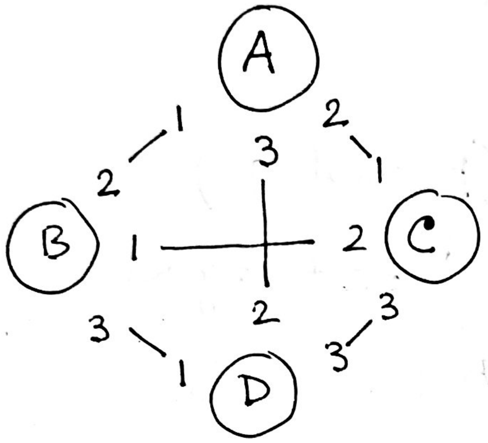
Stable Roommates Problem

Instead of two groups of agents and one group having preferences over the other, consider a single group of agents that are to be paired up. Objective: stability.



A : B C D

B : C A D

C : A B D

D : B A C

AB, CD → BC blocks

BD, AC → AB blocks

AD, BC → AC blocks

Algorithm to find stable roommates

~~Part 1~~ Phase 1

A : $\boxed{\cancel{C}}$ B $\cancel{D}$

B : $\boxed{A}$ $\cancel{C}$ $\cancel{D}$

C : $\boxed{D}$ $\cancel{B}$ $\cancel{A}$

D : $\boxed{\cancel{B}}$ $\cancel{A}$ C

- Each agent approaches his/her favorite roommate
- If a roommate gets two offers, it keeps the best and rejects the other (eliminate in pairs)
- The rejected agent approaches his next most preferred
- Phase 1 ends when no agent has any more offers to make

If an agents all options are eliminated, no stable roommate exists

If Phase 1 survives all agents, then a stable table is obtained

~~Phase 2~~: Cleaning up: all agents below an agents existing offers are eliminated.

Phase 2:

Make another table (p and q)

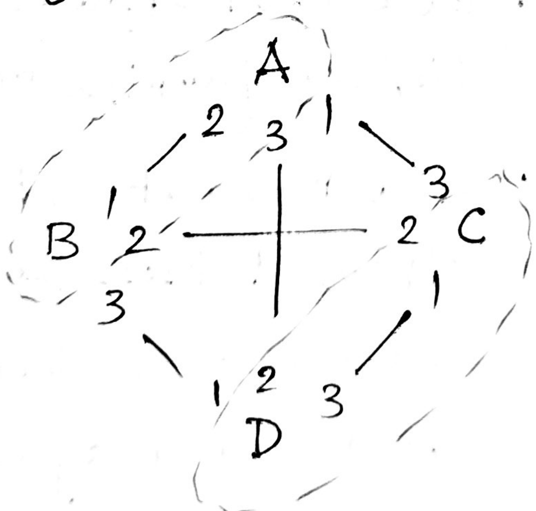
- p: last agent ; q: 2nd best agent

p : A D A   p : B B

q : B C   q : C

- if any cycle found in p, eliminate it in the way q_i reject p_{i+1} (eliminate in pairs)
- clean up the table and continue with any agent that has at least 2 agents left in its preference.

(AB, CD) final match



Irving's Algorithm

Irving 1985:

An efficient algorithm for stable roommates problem.

Journal of Algorithms.

Strategyproofness in ~~stable~~ matching

A matching X ~~fails~~ is manipulable ~~strategyproofness~~ if $\exists$ ~~some~~ $i \in N$ and a profile P s.t.

$$X(P_i', P_{-i}) \; P_i \; X(P_i, P_{-i}).$$

Matching is strategyproof if it ~~does~~ is not ~~fail strategyproofness~~ manipulable on any profile for any player.

i.e., in every profile every player either gets the same matching or a worse matching by manipulating.

What about DA?

Thm: DA algorithm is ~~not strategy~~ manipulable.

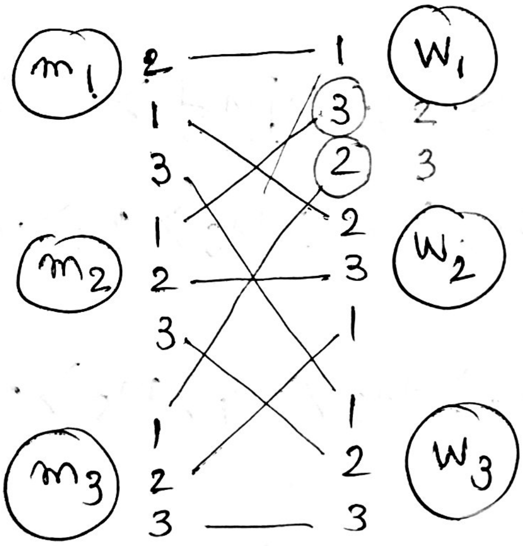
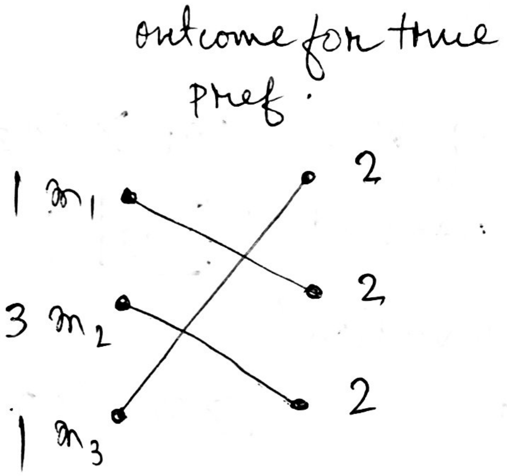
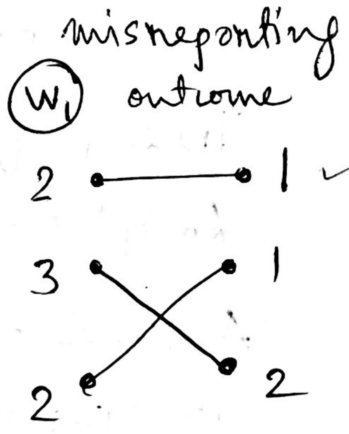


Fig. 1





player w_1 gets better off.

men-proposing DA is manipulable by women.

Thm: men-proposing DA is strategyproof by men.

Proof: See the proof in Dubins & Freedman (1981).

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The women in men-proposing (men in women-proposing) DA is manipulable

How hard is it to manipulate? Easily.

Use a structural result from optimal manipulation.

Claim: An optimal manipulation by a woman in mp-DA can be found in polynomial time.

Optimal: Among all possible manipulation a woman can perform, the one that gives her best man in mp-DA w.r.t. her own preference.

Structural result: If an optimal misreport matches her with a man through some misreport, that matching can be reached via a "single-elevation" misreport.

True pref. of w: $m_1 > m_2 > m_3 > m_4 > \boxed{m_5} > m_6 > m_7 > m_8$

An optimal manipulation: $\boxed{m_2} > m_4 > m_1 > m_8 > m_6 > \dots > m_7$

A "single-elevation" misreport: $m_1 > m_2 > m_6 > m_3 > m_4 > m_5 > m_7 > m_8$

Thm: Any optimal manipulation for a woman in mp-DA can also be achieved via a 'single-elevation' misreport. [Vaish and Garg IJCAI 2017]

Q: What is the optimal manipulation algorithm?

A: Find all possible single elevations ($O(n^2)$) and compute DA for each of them.

Stability question after manipulation?

Thm: The DA matching after optimal manipulation by a woman is stable w.r.t. the true preferences.

Obs 1: Consider a single elevation misreport of woman w. – she elevates m.

If m proposed w in the original profile P then m will propose w in the misreported profile P'_w, P_{-w}

- m proposed w in the original profile because it got rejected by all the women that are ranked above w by m (call this set $S_m^{\text{above } w}$ -- the set could be empty).
- by elevating m in her preference in the manipulated profile, now w is potentially rejecting some men that she was tentatively accepting earlier.
- these rejected men may propose to the women in set $S_m^{\text{above } w}$, which is only *increasing* the number of men proposing to those women. Note, the men who are not moved down below m by w in the manipulated profile will have the execution to be the same as before.
- hence, if m has been rejected by all the women in $S_m^{\text{above } w}$ earlier, it will continue to be rejected by them, resulting in m proposing w in the manipulated profile as well.

P = true profile

$X = DA(P)$

$P' = P'_w, P_{-w}$ = manipulated profile

$X' = DA(P')$

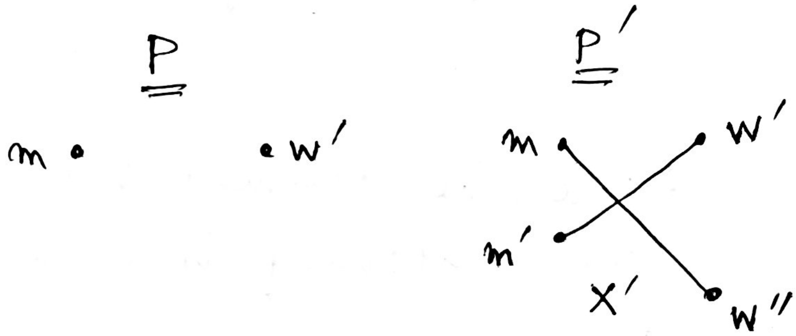
Suppose not, X' is not stable w.r.t. P (original profile)

let (m, w') blocks X' in P

$\boxed{\text{Claim:}}$ $w' = w$

If not, then w' and m prefers each other than what they have been matched to in X'

Since they are blocking pair in P,



$m \; P_{w'} \; m' = X'(w')$

$w' \; P_m \; w'' = X'(m)$

But in these two profiles P_m and $P_{w'}$ hasn't changed.

Then (m, w') is also a blocking pair of X' under P' violates the stability of DA.

Hence (m, w) blocks X' w.r.t. P.

Construct

P_m	P_w	P'_m	P'_w	P''_m	P''_w
					m
					$\vdots$
w	m	w	$X'(w)$	w	$X'(w)$
					$\vdots$
$X'(m)$	$X'(w)$	$X'(m)$	m	$X'(m)$	

$P''_w \rightarrow m$ moves to the top, all other preferences are similar to P'_w moved one level below.

- m must propose to w in $DA(P')$
  since m is matched to $X'(m)$ below w

$\Rightarrow$ • m must propose to w in $DA(P'')$
  from Obs 1.

$\Rightarrow$ • $X''(w) = m \rightarrow X'' = DA(P'')$
  because m is the top choice of w in P''.

But P''_w gives a better partner ~~under~~ (acc. to P) than her optimal manipulation!
compare m's position in P and P''. $\square$

Hence, the optimal manipulation men-optimal ~~out~~ matching is a member of the original lattice.

[stable matchings are manipulable, but optimally manipulated matching is stable]