Bargaining Games

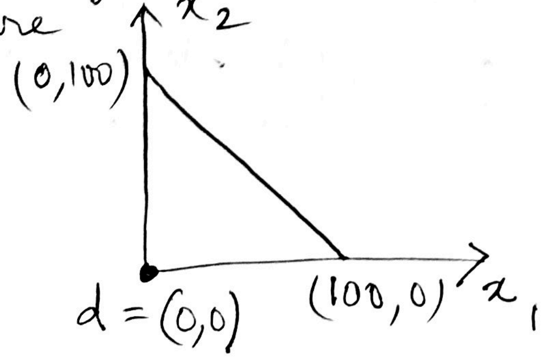
So far we have analyzed only non-cooperative games, where agents cannot communicate with each other. However, we have seen situations where taking decisions collectively may be better. Recall: the ideas of correlated equilibria in games, where strategies are defined over an action profile, rather than on individual actions.

- The model of game theory, that deals with agents collectively, is called "cooperative games".
- Note: this model only opens up the option of communicating with each other. The agent still remain self-interested, i.e., they still want to maximize their own reward.

The first model to consider in cooperative games is the "bargaining" model.

Setting: A set of possible outcomes are bargained on, and finally certain outcomes are recommended to the players by an arbitrator (trusted ~~agent~~ third party)

Example: 2 players divide ₹100 among them.
If their bargain is successful, they divide the money accordingly, otherwise, none gets anything. The failure to reach an agreement is denoted as a disagreement point, $d = (0,0)$. The possible allocations are



If the value of money for each agent is equal to the money itself Then the set

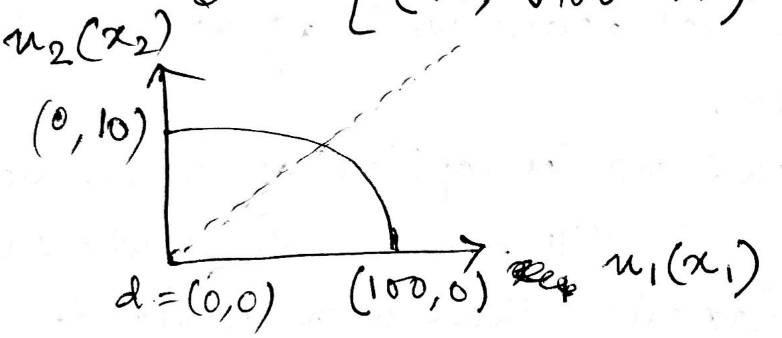
$$S = \{(x, 100-x) : 0 \leq x \leq 100\}$$

denotes the utility space of all possible allocations.

It is also reasonable to assume that the money be equally split among them.

Suppose agent 1's value for money is $u_1(x) = x$ and agent 2's: $u_2(x) = \sqrt{x}$. Then the utility space for all allocations

$$\hat{S} = \{(x, \sqrt{100-x}) : 0 \leq x \leq 1\}$$



In this case a reasonable division would lead to equal utility and not equal money.

Model: Bargaining between two agents using a set $S \subseteq \mathbb{R}^2$ – set of feasible allocations and a vector $d \in \mathbb{R}^2$ – disagreement point.

A bargaining problem instance is the tuple (S, d). A solution concept should find a point in S that satisfies a set of desirable properties – axiomatic approach.

Notation for vectors, say $x, y \in \mathbb{R}^n$

$x \geq y \Rightarrow x_i \geq y_i \ \forall i$, $x_j > y_j$ for some j.

$x \gg y \Rightarrow x_i > y_i \ \forall i$.

$xy = (x_i y_i, \ i \in \{1, \ldots, m\})$ elementwise product.

Bargaining game [some additional details]

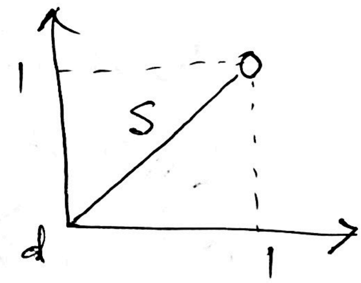
- is an ordered pair (S, d), $S \subseteq \mathbb{R}^2$, $d \in \mathbb{R}^2$
- S is a nonempty, compact, and convex set — the set of alternatives.
- $d = (d_1, d_2)$ is the disagreement point
- $\exists$ ~~at lo~~ $x = (x_1, x_2) \in S$ satisfying $x \gg d$.

Collection of all bargaining games ~~as~~ is denoted by $\mathcal{F}$.

Why these assumptions on the set S?

Compact = closed + bounded in $\mathbb{R}^m$

- closed: s.t. all sequences have a limit point within the set. In the example, each point has a better point in the set S, but the limit of that sequence is not in S. $S = \{(x, x) : 0 \leq x < 1\}$



- bounded: ~~ones~~ objective / players' objectives are to maximize their payoffs from this set. But that maximal ~~point~~ value needs to be bounded.
- Convex: weighted average of possible alternatives is also an alternative.

  e.g., a lottery that chooses one possible outcome w.p. p and another w.p. $(1-p)$ should be possible to achieve via the bargaining process.
- $\exists x \in S$, s.t. $x \gg d$: to avoid degenerate solutions.

Solution concept in bargaining game:

$$\phi : \mathcal{F} \to \mathbb{R}^2 \quad \text{s.t.} \quad \phi(S, d) \in S \quad \text{for each game } (S, d) \in \mathcal{F}$$

~~set of all bargaining games~~

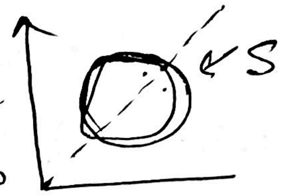
Desirable properties

① Symmetry:

- A bargaining game (S,d) is symmetric if

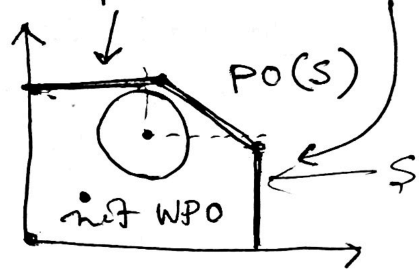
  (a) $d_1 = d_2$

  (b) if $x = (x_1, x_2) \in S$, then $(x_2, x_1) \in S$



Defn: A solution concept ϕ is symmetric if for every symmetric bargaining game (S,d), $\phi(S,d)$ is s.t. $\phi_1(S,d) = \phi_2(S,d)$

② Efficiency:



An alternative $x \in S$ is an efficient point if $\nexists\, y \in S$, $y \neq x$ s.t. $y \geq x$ [all agents weakly prefer y to x and at least one agent strictly]

- An alternative $z \in S$ is weakly efficient if $\nexists\, y \in S$, $y \neq z$ s.t. $y \gg z$. [all agents strictly prefer y to z]

  $PO(S)$: set of all Pareto optimal (efficient) points

Defn: A solution concept ϕ is efficient if $\phi(S,d) \in PO(S)$ for every bargaining game $(S,d) \in \mathcal{F}$.

③ Covariance under positive affine transformation

motivation: the bargaining solution should be scale-free – independent of the units of utility
also, should be affected in the same way a translation is introduced to the possible allocations.

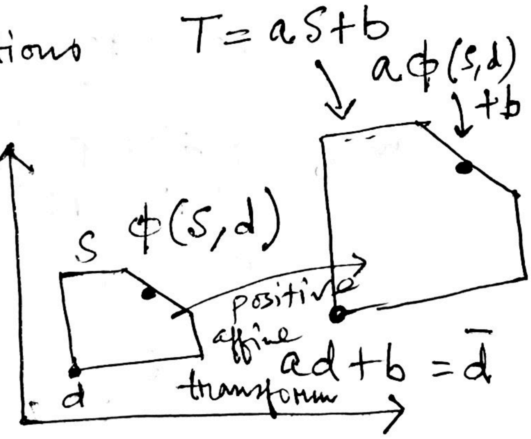
$$aS + b = \{(as+b) : s \in S\}$$
$$= \{(a_1 s_1 + b_1 , a_2 s_2 + b_2) : (s_1, s_2) \in S\}$$

similarly $ad + b = (a_1 d_1 + b_1 , a_2 d_2 + b_2)$.

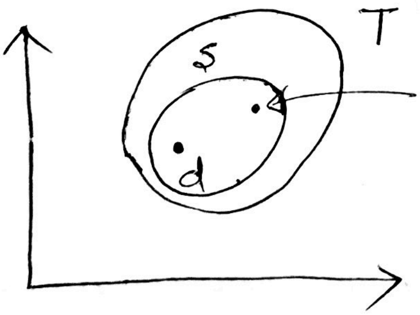
Defn: A solution concept ϕ is covariant under positive affine transformations if for every bargaining game $(S,d) \in \mathcal{F}$, for every $a \in \mathbb{R}^2$, $a \gg 0$, and $b \in \mathbb{R}^2$

$$\phi(aS + b, ad + b) = a\phi(S,d) + b$$

transform set of feasible allocations and disagreement points



④ Independence of Irrelevant Alternatives



$S \subseteq T$

$\phi(T,d)$

What should $\phi(S,d)$ be?

Will be strange if $\phi(S,d)$ is not the same, since that option was available in T.

Defn: A solution concept ϕ satisfies IIA if for every bargaining game $(T,d) \in \mathcal{F}$ and for every $S \subseteq T$

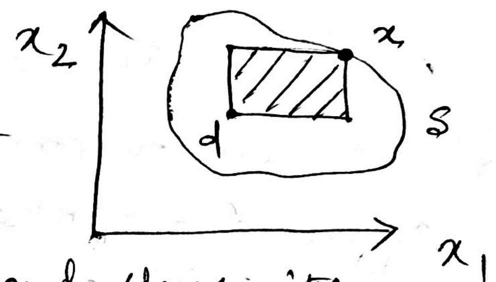
$$\phi(T,d) \in S \Rightarrow \phi(S,d) = \phi(T,d).$$

The Nash solution

Thm: There exists a unique solution concept N for the family of bargaining games $\mathcal{F}$ satisfying symmetry, efficiency, IIA, covariance under positive affine transformations.

$$N(S,d) = \underset{x \in S,\ x \geq d}{\operatorname{argmax}} \ (x_1 - d_1)(x_2 - d_2)$$

— The x that maximizes the area of the axis-parallel rectangle with ~~that~~ left-~~to~~ bottom corner as d and stays within S.



Recall, S is convex, compact, and has at least one "better" point than d.

Proof in three parts:

a) the $N(S,d)$ point is unique

b) $N(S,d)$ satisfies the four properties

c) Any solution concept that satisfies the four properties must be identical to $N(S,d)$

<u>Lemma 1</u>: For every bargaining game ~~there~~ (S,d), there exists a unique point in the set $N(S,d)$.

Proof: Suppose not, then we show we can construct a point that improves the Nash product, contradiction.

First, do a coordinate transformation by adding $-d$ to all points in (S,d), i.e., the new game is $(S-d, (0,0))$

The Nash product is therefore

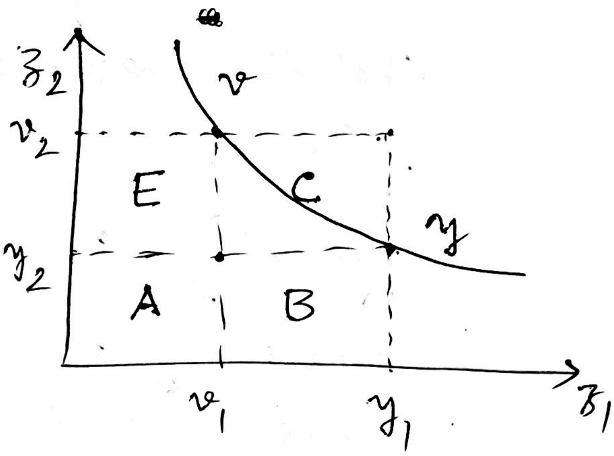
$$\underset{\{z \in S-d,\ z \geq 0\}}{\operatorname{argmax}} \ z_1 z_2$$

Note: The value of the product is unchanged due to the coordinate transform. ~~Let~~ Let $f(z) = z_1 z_2$.
This is a ~~cont~~ continuous function and the domain on which it is maximized, $D := \{z \in S-d,\ z \geq 0\}$ is compact and convex. Also $D \neq \phi$ by assumption of S.
Hence, a maxima is guaranteed to exist.

Suppose, it is not unique i.e.

$$c^* = y_1 y_2 = v_1 v_2$$

both gives rise to the same maximum value of the product.



consider a new point

$$w = \frac{1}{2} y + \frac{1}{2} v$$

$w \in D$, because of convexity

Then
$$w_1 w_2 = \left(\frac{1}{2} y_1 + \frac{1}{2} v_1\right)\left(\frac{1}{2} y_2 + \frac{1}{2} v_2\right)$$

$$= \frac{1}{4} y_1 y_2 + \frac{1}{4} v_1 v_2 + \frac{1}{4}\left(y_1 v_2 + v_1 y_2\right)$$

$$\left[\begin{array}{l} y_1 v_2 + v_1 y_2 = A + B + C + E + A = 2A + B + C + E \\ y_1 y_2 + v_1 v_2 = A + E + A + B = 2A + B + E \end{array}\right]$$

$$> \frac{1}{4}\left(y_1 y_2 + v_1 v_2\right) + \frac{1}{4}\left(y_1 y_2 + v_1 v_2\right)$$

$$= c^* \quad \rightarrow\leftarrow$$

∎

12-8.

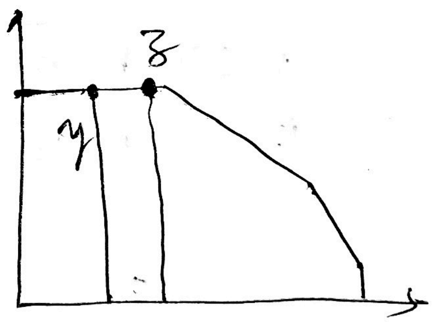
Lemma 2: $N(S,d)$ satisfies symmetry, efficiency, covariance under positive affine transformations, and IIA.

Proof: (Symmetry) ~~Suppose d~~ given $d_1 = d_2 = d$ and S is symmetric. Suppose y^* maximizes

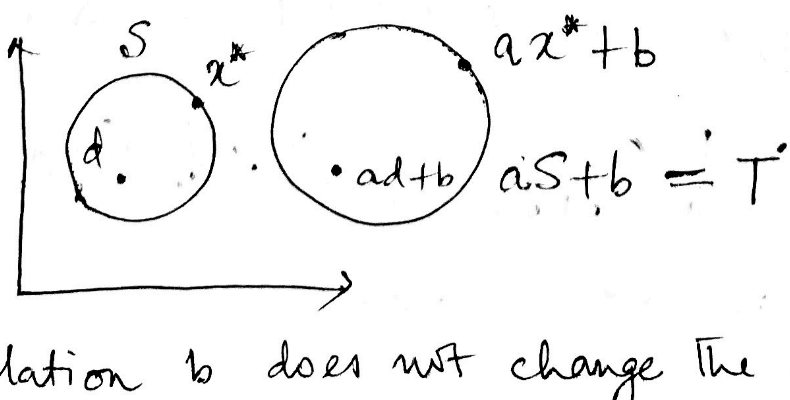
$$(y_1 - d)(y_2 - d)$$

Then $y^* = (y_1^*, y_2^*)$ and $z = (y_2^*, y_1^*) \in S$ also maximizes the product. Since we know that the maxima has to be unique, then $y_1^* = y_2^*$.

(efficient) Suppose not, if z is s.t. $z \geq y$ where y is the optimal argument for the Nash product. But then $(z_1 - d_1)(z_2 - d_2)$ strictly improves the area of the rectangle/Nash product. Contradicts that y is Nash optimal.



(CPAT) Suppose $x^* = N(S,d)$ is the Nash optimal solution. Consider $aS + b$, where $a >> 0$.



translation b does not change the area of a rectangle.

modified objective function

$$\underset{s_1, s_2}{\arg\max} \left((a_1 s_1 + b_1) - (a_1 d_1 + b_1)\right)\left((a_2 s_2 + b_2) - (a_2 d_2 + b_2)\right)$$

$$= \underset{s_1, s_2}{\arg\max}\ a_1 a_2 (s_1 - d_1)(s_2 - d_2) = x^*$$

Hence, the optimal solution in T is $ax^* + b$.

(IIA) Straightforward since if a maxima of a function over a larger set stays in a smaller set, that continues to be the optimal even in the smaller set.

Lemma 3: Every solution concept ϕ satisfying symmetry, efficiency, CPAT, and IIA is identical to N.

Proof idea: Use PAT to move d to (0,0) and the ~~optimal~~ Nash optimal point $y^* = N(S,d)$ to (1,1).
- Use the 4 properties to show that $\phi(S,d)$ that satisfies these 4 must be y^*.

Step 1: Since $\exists$ at least one $x \in S$ s.t. $x \gg d$

$y^* \gg d$

$$L(x_1, x_2) = \left(\frac{x_1 - d_1}{y_1^* - d_1}, \frac{x_2 - d_2}{y_2^* - d_2}\right), \quad x \in S.$$

clearly $L(d_1, d_2) = (0,0)$

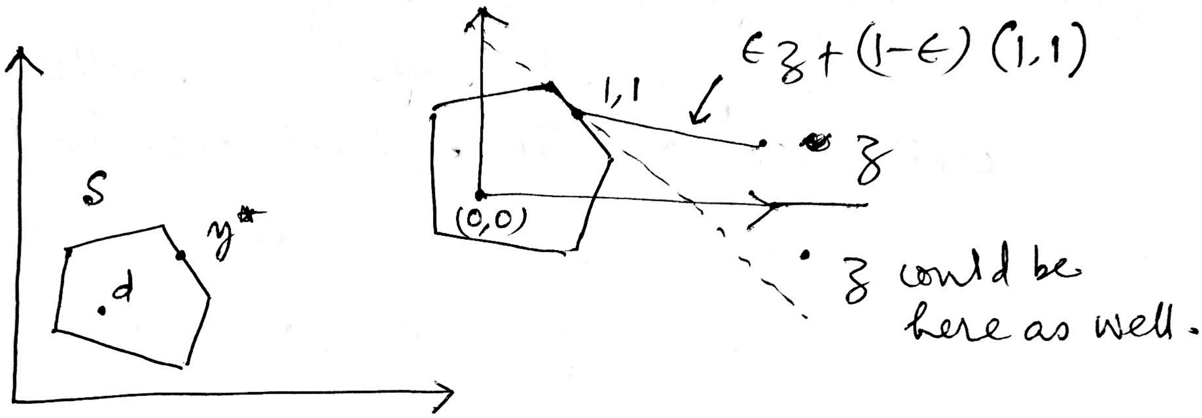
$L(y_1^*, y_2^*) = (1,1)$

Step 2: $x_1 + x_2 \leq 2 \quad \forall \ x \in aS+b$.

Suppose not, say $\exists$ some $z \in L$ s.t. $z_1 + z_2 > 2$

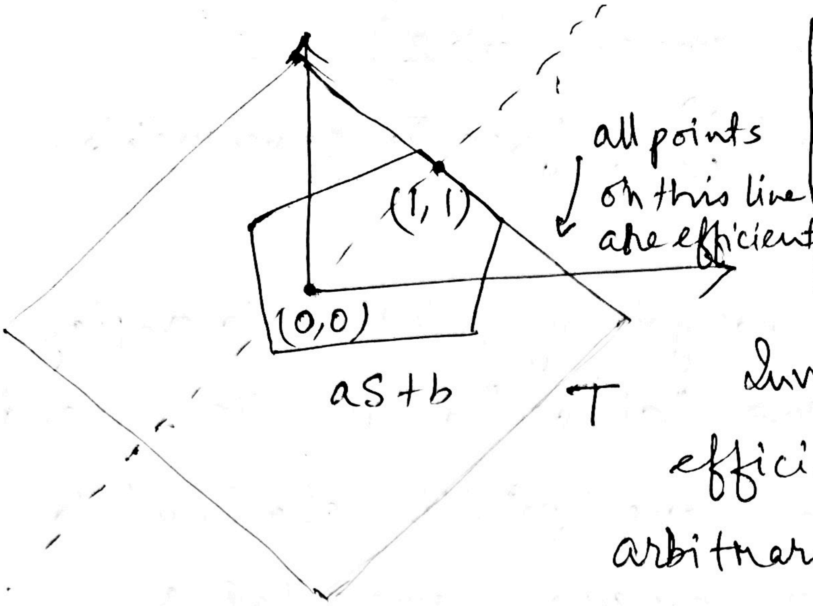
We know if y^* maximizes the Nash product in original domain, $(1,1)$ maximizes the Nash product $y_1 y_2$ in the new domain L.

Since S was convex, L will also be.



consider a point $(1-\epsilon)(1,1) + \epsilon z = (l_1, l_2)$ for some $\epsilon > 0$ ~~suffienct~~ sufficiently close to 0, the product $l_1 l_2 > 1$. This is a contradiction since (l_1, l_2) gives a larger Nash product than the maxima.

Step 3: Enclose $aS+b$ with a square symmetric along ~~the~~ $y_1 = y_2$ line and one side along the $y_1 + y_2 = 2$ line



This is possible since $aS+b$ is bounded

Invoke symmetry and efficiency for any arbitrary solution concept ϕ

$$\phi(T,(0,0)) = (1,1)$$

Now, ϕ also satisfies IIA. $aS+b \subseteq T$ and contains $(1,1)$, hence $\phi(aS+b,(0,0))=(1,1)$

ϕ satisfies CPAT, apply L^{-1} (possible since all a_i's are positive)

$$\begin{cases} L(S,d) = aS+b \\ L(y^*) = (1,1) \end{cases}$$

This gives $\phi(S,d) = y^*$

$= N(S,d)$

$\square$