Solution concepts for TU games

- Agents are given monetary transfers
- ~~An~~ Agents in a ~~~~ coalition can at most get what they earn.

Say, $i \in S$ and the value of S is $v(S)$

then $\sum_{i \in N} y_i \leq v(S)$ is possible

This is the coalitional threat that a coalition can propose.

Coalitional Rationality

A share of valuations $x \in \mathbb{R}^n$ [also called imputation] is coalitionally rational

if $\sum_{i \in S} x_i \geq v(S)$. $\forall S \subseteq N$.

[the coalition is happy with the division]

However, $\sum_{i \in N} x_i = v(N)$ [definition of imputation]

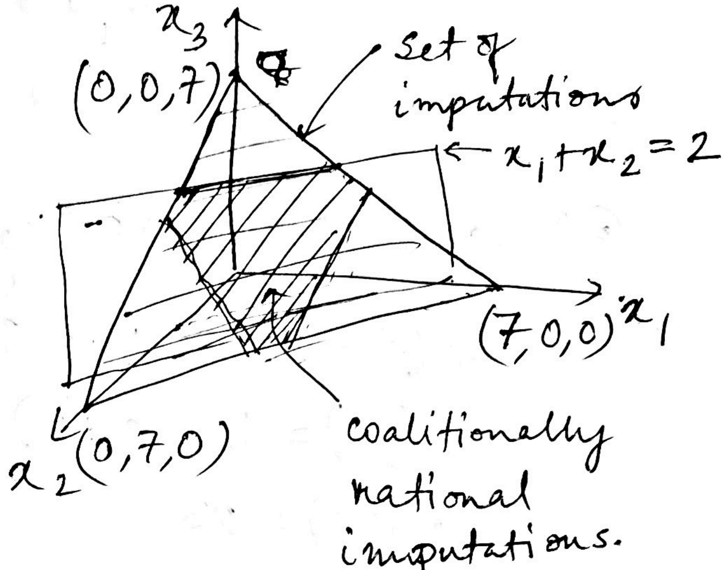
[The sum payment to the agents can't be more than the value of the grand coalition]

Remark: An imputation x is individually rational

if $x_i \geq v(\{i\})$

<u>Example</u>: $N=\{1,2,3\}$

$$v(1)=v(2)=v(3)=0, \quad v(12)=2, \quad v(1,3)=3$$
$$v(2,3)=4, \quad v(1,2,3)=7$$



<u>Core</u>! An imputation x is in core if it ~~satisfies~~ is coalitionally rational

i.e.,

(1) $\sum_{i\in S} x_i \geq v(S) \quad \forall S \subsetneq N$

(2) $\sum_{i\in N} x_i = v(N)$.

Check the cores of the previous examples

(1) DTM-ver1:

$$C(N,v)=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1+x_2+x_3=300, \; x_1,x_2,x_3\geq 0\}$$

(2) DTM.v2:

$$C(N,v)=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1+x_2+x_3=300, \; x_1+x_2\geq 300, \; x_1,x_2,x_3\geq 0\}$$
$$=\{x\in\mathbb{R}^3 : x_1+x_2=300, \; x_1,x_2\geq 0, \; x_3=0\}$$

(3) DTM.v3:

$$x_1+x_2+x_3=300$$
$$x_1+x_2 \geq 300$$
$$x_1+x_3 \geq 300$$
$$x_2,x_3\geq 0$$

$$C(N,v)=\{x\in\mathbb{R}^3 : x_1=300, \; x_2=0, \; x_3=0\}$$

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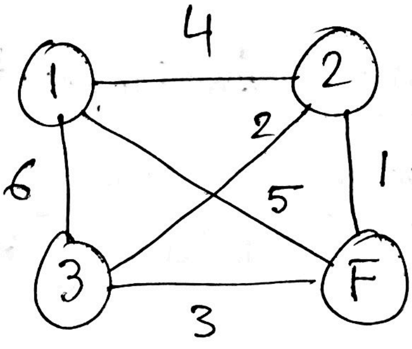
(4) DTM.v4 :

$$x_1+x_2+x_3=300$$
$$x_1+x_2 \geq 300$$
$$x_1+x_3 \geq 300$$
$$x_2+x_3 \geq 300$$

no solution

$C(N,v)=\phi$.

(5) Core of the MST game



$v(1)=5,\ v(2)=9,\ v(3)=7$

$v(12)=15,\ v(13)=12,\ v(23)=17$

$v(123)=23$

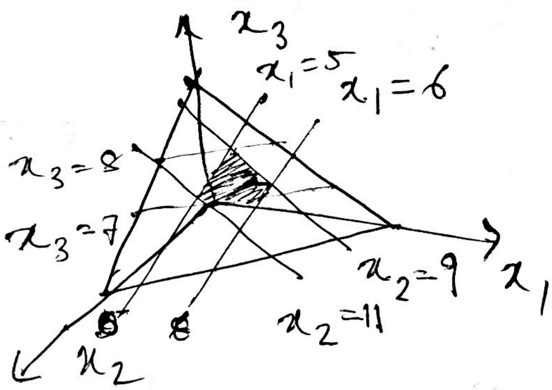
$x_1 \geq 5,\ x_2 \geq 9,\ x_3 \geq 7$

$x_1+x_2+x_3=23$

$x_1+x_2 \geq 15 \Rightarrow x_3 \leq 8$

$x_2+x_3 \geq 17 \Rightarrow x_1 \leq 6$

$x_3+x_1 \geq 12 \Rightarrow x_2 \leq 11$



Core is non-empty.

(6) Core of Bankruptcy game : Homework.

Core can be empty. How to know if core is non-empty?

~~Theorem~~ (~~Bondareva-Shapley~~ '63, '67)

<u>Balanced weights</u>: A set of non-negative weights (over 2^N, all coalitions) λ is balanced if

$$\sum_{\{S\subseteq N:\, i\in S\}} \lambda(S) = 1, \quad \forall i \in N.$$

<u>Example</u>: $N = \{1, 2, 3\}$

$S = \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$

$\lambda(\{1,2\}) = \lambda(\{2,3\}) = \lambda(\{1,3\}) = \frac{1}{2}$.

Bondareva - Shapley theorem ('63, '67)

A ~~coalitional~~ TU game (N, v) has a non-empty core iff for <u>every</u> balanced ~~coalition~~ set of weights λ,

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) \, v(S)$$

<u>Proof</u>: Calculation of core.

$$\min \sum_{i \in N} x_i \quad \cdots (1)$$

$$\text{s.t.} \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N$$

If the optimal solution x^* be s.t.

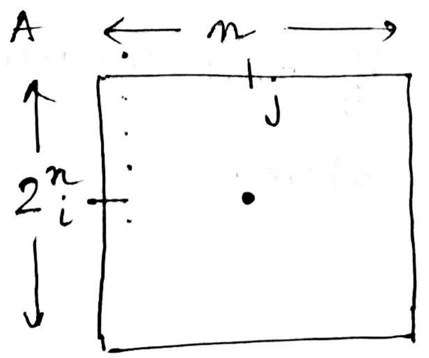
$$\sum_{i \in N} x_i^* > v(N) \Leftrightarrow \text{core is empty.}$$

$$\Rightarrow \sum_{i \in N} x_i^* = v(N) \Leftrightarrow \text{core is non-empty.}$$

Dual of (1)

$\min c^T x$

$A x \geq b$



$A(i,j) = \begin{cases} 1, & \text{if } j \in i^{th} \text{ coalition} \\ 0, & \text{ow} \end{cases}$

$b = \begin{bmatrix} v(\phi) \\ v(1) \\ \vdots \\ v(N) \end{bmatrix}$

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dual = $\max \ \lambda^T b$

$$\text{s.t.} \quad \lambda^T A = c^T$$
$$\lambda \geq 0$$

$$\Rightarrow \quad \max \sum_{S \subseteq N} \lambda(S)\, v(S) \qquad \text{-- (2)}$$
$$\text{s.t.} \sum_{S \subseteq N : i \in N} \lambda(S) = 1$$
$$\lambda(S) \geq 0$$

$$\sum_{S \subseteq N} \lambda(S)\, v(S) \leq \sum_{i \in N} x_i^* = v(N) \Leftrightarrow \text{core is non-empty.}$$

$\forall \lambda$ satisfying the constraints of (2) = balanced weights. $\square$

Ex. DTM.v4. $v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 300 = v(\{1,2,3\})$.

$$\lambda(12) = \lambda(23) = \lambda(13) = \frac{1}{2}$$

$$\sum_{S \subseteq N} \lambda(S)\, v(S) = \frac{1}{2} \times 300 \times 3 > 300 = v(N).$$

Easy to check if this condition is satisfied or not. Solve the dual above.

Convex games:

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D) \quad \forall C, D \subseteq N.$$

$$\Leftrightarrow \quad v(A \cup \{i\}) - v(A) \leq v(B \cup \{i\}) - v(B) \quad \forall A \subseteq B \subseteq N$$
$$\forall i \in N \setminus B.$$

Proved last time

Claim: Convex games have non-empty core.

Proof: Using B-S characterization. — Approach ①

$$x_1 = v(1)$$
$$x_2 = v(12) - v(1)$$
$$x_3 = v(123) - v(12)$$
$$\vdots$$
$$x_n = v(N) - v(1, \ldots, n-1)$$

Why: $\forall S \subseteq N \quad \sum_{i \in S} x_i \geq v(S)$

Clearly, $\sum_{i \in N} x_i = v(N)$

Pick arbitrary $S = \{i_1, i_2, \ldots, i_k\}$ → in the lexicographic order

$$x_{i_1} = v(1, \ldots, i_1) - v(1, \ldots, i_1 - 1) \geq v(i_1) - v(\phi)$$
$$x_{i_2} = v(1, \ldots, i_1, \ldots, i_2) - v(1, \ldots, i_2 - 1) \geq v(i_1 i_2) - v(i_1)$$
$$\vdots$$
$$x_{i_k} = v(1, \ldots, i_1, \ldots, i_2, \ldots, i_{k-1}, \ldots, i_k) - v(1, \ldots, i_k - 1)$$
$$\geq v(i_1 i_2 \ldots, i_k) - v(i_1 \ldots, i_{k-1})$$

$$\sum_{l=1}^{k} x_{i_l} = \sum_{i \in S} x_i \geq v(S). \quad \text{(Proved)}$$