Productive Output in Hierarchical Crowdsourcing

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ABSTRACT

Organically grown crowdsourcing networks, which includes production firms and social network-based crowdsourcing applications, tend to have a hierarchical structure. Considering the entire crowdsourcing system as a consolidated organization, a primary goal of a designer is to maximize the net productive output of this hierarchy using reward sharing as an incentive tool. Every individual in a hierarchy has a limited amount of effort that they can split between production and communication. Productive effort yields an agent a direct payoff, while the communication effort of an agent improves the productivity of other agents in her subtree. To understand how the net output of the crowdsourcing network is influenced by these components, we develop a game theoretic model that helps explain how the individuals trade off these two components depending on their position in the hierarchy and their shares of reward. We provide a detailed analysis of the Nash equilibrium efforts and a design recipe of the reward sharing scheme that maximizes the net productive output. Our results show that even under strategic behavior of the agents, it is sometimes possible to achieve the optimal output and also provide bounds on the achievability when this is not the case.

Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics; K.4.3 [**Computers and Society**]: Organizational Impacts

Keywords

Hierarchies; Crowdsourcing; Nash Equilibrium; Social Output; Price of Anarchy

1. INTRODUCTION

The organization of economic activity as a means for the efficient co-ordination of effort is a cornerstone of economic theory. We take the perspective that organizations have the goal of 'crowdsourcing' production or other economic activity through incentives, to maximize production at minimum cost. Organizations that grow over time, either through referrals or hiring, tend to have a hierarchical structure. In addition to typical large corporations, more recent examples of hierarchical organizations include those that arise in 'diffusion-based task environments' where agents become aware of tasks through recruitment [18, 25]. A well known example of this is the winning solution of the 2009 DARPA Red Balloon Challenge, who adopted an indirect reward scheme where the reward associated with successful completion of subtasks was shared with other agents in the network [18]. This example gave rise to its analysis in the context of identity fraud [16] or in information misreporting and verification costs [15]. In this context, our study serves as a complementary approach of understanding how individuals trade off efforts between *searching* and *spreading the message* that lead to the productive output of an already formed hierarchy.

There is a long history on the role of organizational structure on economic efficiency [23]. More recently, [19, 20, 14] study the role of hierarchies; see [24] for a survey of different perspectives. In this paper, we draw attention to the *interaction* between various common aspects of network influence, such as profit sharing [10], information exchange [4], influence and production in crowdsourcing networks. At the same time different individuals in a network exert different amounts of effort toward various tasks. In this paper, we are motivated by the possibility that the phenomenon can be understood as a consequence of the strategic behavior of the participants, the *reward sharing scheme* and their *positions in the network*.

In networked organizations, agents are responsible for two processes: information flow (communication effort) and task execution (productive effort). The objective of the organization designer is to maximize the net productive output of the networked system. However, the individuals in an organization are rational and intelligent and select the level of effort which maximizes their payoff. Hence, to understand how organizations can boost their productive output, we need to understand how the individuals connected over a network split their efforts between work *vs.* investing effort in communicating tasks to others depending on the amount of direct and indirect rewards. When an agent communicates with another, we call the former an *influencer* and the latter an *influencee*. Influencers can improve the productivity of the influencees, at the cost of reducing their own production. Influencees, in turn, share a part of their rewards with the influencers, and this interaction induces a game between the agents connected over the network.

1.1 Overview and Main Results

We model the network as a directed graph, where the di-

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rection represents the direction of information flow or communication between nodes and the rewards are shared in the reverse direction. For an easier exposition, in this work, we focus on hierarchies where the network is a directed tree. Each agent in the organization decides how to split its effort between (i) production effort, which incurs a cost to the agent but results in direct payoff and indirect reward to other agents on the path from the root to the agent, and (ii) communication effort, which serves to improve the productivity of his descendants on the tree (e.g., explaining the problem to others, conveying insights and the goals of the organization). Committing effort to communication can improve productivity of descendants, which in turn improves their output, should they decide to invest effort in direct work, and thus give an agent a return on investment through an indirect payoff. A natural constraint is imposed on the total effort spent in complementary tasks of production and communication.

Each agent decides, *based on his position in the hierarchy*, how to split his effort between production and communication, in order to maximize the sum of direct payoff and indirect reward, accounting for the cost of effort. For most of our results we adopt a specific *exponential productivity* (EP) model, where the quality of communication falls exponentially with effort spent in production with a parameter β. The model has the useful property that a pure-strategy Nash equilibrium always exists and has a closed form expression (Theorem 1). We develop tight conditions for the uniqueness of the equilibrium (Theorems 2 and 3).

Based on these results, we are then able to ask and answer the question *'What effect does the design of reward share have to maximize the social output of a hierarchical organization?'* We define the *social output* to be the sum of the individual outputs which are products of *productivity*, due to the communication efforts of ancestors, and individual *production effort*. Our next result is that for *balanced hierarchies* with EP, there exists a threshold β^* on a communication quality parameter β such that if $\beta \leq \beta^*$, i.e., communication is 'good enough', then the *equilibrium social output* can be made equal to the *optimal social output* by choosing *an appropriate* reward sharing scheme. The phenomenon is captured by the fraction called Price of Anarchy (PoA) [13]. If the reward share is not chosen appropriately, PoA can be large (Theorem 4). For $\beta > \beta^*$, i.e., low quality communication, we give closed-form bounds on the PoA (Theorem 5), which we show are tight in special networks. Our results highlight the importance of the design of reward sharing in organizations accounting for both network structure and communication process in order to achieve a higher social output.

1.2 Prior Work

In this section, we describe the literature that is relevant for presenting our results. A complete survey of the literature in organizational theory can be found in [24, 12, 17, 6]. The study of effort levels in network games, where an agent's utility depends on actions of neighboring agents has recently received much attention [8]. For example, [3] show how the level of activity of a given agent depends on the Bonacich centrality of the agent in the network, for a specific utility structure that results in a concave game. Our model differs in two aspects: (a) we have multiple types of efforts (namely production and communication) and (b) we show results for utilities that are non-concave, both of which result in a different structure and form to the correlation among the efforts of agents. In particular, we also provide a specific grounding of our more general results, to exponential decrease in influence and balanced hierarchical organizations, that allows us to derive structural properties of the effect of parameters like communication strength or effectiveness on effort levels of agents. Even for the case of non-linear influences, our results give a design recipe for the reward sharing schemes that maximize production. We also provide a lower bound on the communication that allows for designing reward schemes to achieve the same productive output as a centralized organization. We also provide sufficiency conditions for uniqueness of the Nash equilibrium.

Rogers [21] analyzes the efficiency of equilibria in two specific types of games (i) 'giving' and (ii) 'taking', where an edge means utility is sent on an edge. A strategic model of effort is discussed in the public goods model of [5], where utility is concave in individual agents' efforts, and the structures of the Nash and stable equilibria are shown. Their model applies to a very specific utility structure where the same benefit of the 'public good' is experienced by all the first level neighbors on a graph. In our model, the individual utilities can be asymmetric, and depend on the efforts and reward shares in multiple levels on the graph. Our utility model cleanly separate the effects of two types of influence, that we term information and incentives, and our analysis is post formation of the network. Also, we study games where agents have continuous actions spaces (their effort levels) and so questions of existence and uniqueness are non-trivial. In addition, we are still able to show that for hierarchical tree structured organizational graphs exploiting the structure of the influence of ancestors or descendants can lead to fast algorithms for computing the effort equilibria. To measure the sub-optimality in output due to the self interested nature of agents, we use the Price of Anarchy (PoA) [13]. In the network contribution games literature, [2] considers a model where an agent's contribution locally benefit the nodes who share an edge with him, and give existence and PoA results for pairwise equilibrium for different contribution functions. The PoA in cooperative network formation is considered in [7], while [22, 9] have considered the question in a selfish network routing context. In our model, the strategies are the efforts of the agents, which distinguishes it from the network formation and selfish routing literature, and we use multiple levels of information and reward sharing and study utilities that are asymmetric even for the neighboring nodes in the network, which distinguishes itself from the network contribution games.

Due to space constraints, we have provided some key proofs and sketches of proofs for the rest.

2. A HIERARCHICAL MODEL OF INFLUENCER AND INFLUENCEE

In this section, we formalize a specific version of the hierarchical network model. Let $N = \{1, 2, \ldots, n\}$ denote a set of agents who are connected over a hierarchy T (see Figure 1). Each node i has a set of *influencers*, whose communication efforts influence his own direct payoff, and a set of *influencees*, whose direct payoffs are influenced by node i. In turn the production efforts of these influencees endow agent i with indirect payoffs. The origin (denoted by node θ) is a

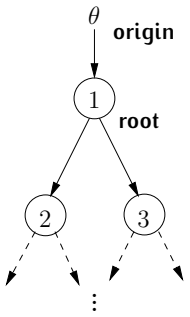

Figure 1: A typical hierarchical model.

node assumed to be outside the network, and communicates perfectly with the first (root) node, denoted by 1.

We number nodes sequentially, so that each child has a higher index than his parent, thus the adjacency matrix is an upper triangular matrix with zeros on the diagonal.

The set of influencers of node i consists of the nodes (excluding node i) on the unique path from the origin to the node, and is denoted by $P_{\theta\to i}$. The set of influencees of node i consists of the nodes (again, excluding node i) in the subtree T_i below her.

The production effort, denoted by $x_i \in [0,1]$, of node i yields a direct payoff to the node, and the particular way in which this occurs depends on its *productivity*. The remaining effort, $1 - x_i$, goes to communication effort, and improves the productivity of the influencees of the node. The constant sum of production effort and communication effort models the constraint on an agent's time, and therefore it is enough to write both the direct and indirect payoff of a node as a function of the production effort x_i. In particular, the productivity of a node, denoted by $p_i(\mathbf{x}_{P_{\theta\to i}})$, depends on the communication effort (and thus the production effort) of the influencers on path $P_{\theta\to i}$ to the node. The production effort profile of these influencers is denoted by $\mathbf{x}_{P_{\theta\to i}}$.

It is useful to associate $x_i p_i(\mathbf{x}_{P_{\theta\to i}})$ with the value from the *direct output* of node i. The payoff to node i comprises of two additive terms that capture: (1) the direct payoff, which depends on the value generated by the direct output of a node and the cost of production and communication effort, and is modulated by the productivity of the node, and (2) the indirect payoff, which is a fraction of the value associated with the direct output of any influencee j of the node. Taken together, the payoff to a single node i is:

$$u_i(x_i, x_{-i}) = p_i(\mathbf{x}_{P_{\theta\to i}}) f(x_i) + \sum_{j\in T_i\setminus\{i\}} h_{ij} p_j(\mathbf{x}_{P_{\theta\to j}}) x_j. \quad (1)$$

The first term is the product of the direct payoff and a function $f(x_i)$ (which models production output and cost) and captures the trade-off between direct output and cost of production and communication effort. The second term is the total indirect payoff received by node i due to the output $p_j(\mathbf{x}_{P_{\theta\to j}})x_j$ of its influencees. We insist that the productivity $p_j(\cdot)$ of any node j is non-decreasing in the communication effort of each influencer, and thus non-increasing in the production effort of each influencer, and we require $\frac{\partial}{\partial x_i} p_j(\mathbf{x}_{P_{\theta\to j}}) \le 0$ for all nodes j, where i is an influencer of j. Each node i receives a share h_{ij} of the value of the direct output of influencee j. The model can also capture a setting where an agent can only share output he creates, i.e., the total fraction of the output an agent retains and shares with the influencers is bounded at 1. Let us assume that agent j retains a share s_{jj} and shares s_{ij} with influencers $i \in P_{\theta\to j}$. A budget-balance constraint on the amount of direct value that can be shared requires $\sum_{i\in P_{\theta\to j}\cup\{j\}} s_{ij} \le 1$. Assume that $s_{jj} = \gamma > 0$, for all j, so that each node retains the same fraction γ of its direct output value. Then, the earlier inequality can be written as, $\sum_{i\in P_{\theta\to j}} \frac{s_{ij}}{\gamma} \le \frac{1}{\gamma} - 1$. By now defining $h_{ij} = \frac{s_{ij}}{\gamma}$, then the whole system is scaled by a factor γ. In addition to notational cleanliness, this transformation gives the advantage of not having any upper bound on the $\sum_{i\in P_{\theta\to j}} h_{ij}$, since any finite sum can always be accommodated with a proper choice of γ. Let us call the matrix $H = [h_{ij}]$ containing all the reward shares as the *reward sharing scheme*.

To highlight our results, we focus on a specific form of the payoff model, namely the *Exponential Productivity* (EP) model, which is an instantiation of the direct-payoff function $f(.)$ and the productivity function $p_i(\cdot)$ as follows.

$$f(x_i) = x_i - \frac{x_i^2}{2} - b\frac{(1-x_i)^2}{2}, \quad (2)$$

$$p_i(\mathbf{x}_{P_{\theta\to i}}) = \prod_{k\in P_{\theta\to i}} \mu(C_k) e^{-\beta x_k}, \quad (3)$$

where $b \ge 0$ is the cost of communication, C_k is the number of children of node k, function $\mu(C_k) \in [0,1]$ is required to be non-increasing, and $\beta \ge 0$ denotes the quality of communication, with higher β corresponding to a lower quality of communication. We assume $p_1 = 1$ for the root node to denote that the root gets unattenuated information. We interpret the term $\mu(C_k)e^{-\beta x_k} \in [0,1]$ as the communication influence of node k on the agents in his subtree.

The direct payoff of an agent i is quadratic in production effort x_i, and reflects a linear benefit x_i from direct production effort but a quadratic cost $x_i^2/2$ for effort. The utility model given by Equation (1) resembles the utility model given in [3]. However, there are a few subtle differences in our model than that in this paper: (a) each agent has two types of effort, namely production and communication, and the communication effort of an agent is *complementary* to the production efforts of her influencees, while the production efforts are *substitutable* to each other. Also, the complementarity is nonlinear, which captures a more general form of reward sharing. (b) We also consider the cost of communication, captured by $b(1-x_i)^2/2$. The productivity of node j, given by $p_j(\mathbf{x}_{P_{\theta\to j}})$, where $j \in T_i \setminus \{i\}$ warrants careful observation. Here we explain the components of this function and the reasons for choosing them. Consider $\mu(C_k)$, which is non-increasing in the number of children, C_k, captures the idea that the effect of the communication effort is reduced if the node has more children to communicate with. An increase in production effort x_k reduces the productivity of influencees of node k. In particular, the exponential term in the productivity captures two effects: (a) a linear decrease in production effort gives exponential gain in the productivity of influencee, which captures the importance of communication and management in organizations [1]. Smaller values of β model better communication and a stronger positive effect on an influencee. (b) We can approximate other models by choosing β appropriately, e.g., linear productivity corresponds to small values of β. This property is useful when the effects of production and communication

on the payoff are equally important. For large β there is very small communication quality between agents and the value of communication effort is low.

The successive product of these exponential terms in the path from root to a node reflects the fact that a change in the production effort of an agent affects the productivity of the entire subtree below her. In the next section we will demonstrate the structure and required conditions for uniqueness of a Nash equilibrium. For brevity of notation, we will drop the arguments of productivity p_i at certain places where it is understood.

Our results on the structure, uniqueness of the equilibrium and their interpretations generalize to other network structures beyond hierarchies, which we skip for space limitations. The applications pervade beyond crowdsourcing into more general models of networked organizations. Even though the simplicity of the EP model gives certain analytical tractability, it serves to illustrate the importance of influence, both communication and incentives, and gives insight on outcome efforts in a networked organization.

2.1 Results on the Equilibrium Efforts

The effect of communication efforts between nodes i and j, where $i \in P_{\theta\to j}$ is captured by the fractional productivity $\frac{p_j}{p_i}$ defined as, $p_{ij}(\mathbf{x}_{P_{i_-\to j}}) = \prod_{k\in P_{i_-\to j}} \mu(C_k)e^{-\beta x_k}$, (the node i_- is the parent of i in the hierarchy). This term is dependent only on the production efforts in the path segment between i and j and accounts for 'local' effects. We show in the following theorem that the Nash equilibrium production effort of node i depends on this local information from all its descendants.

THEOREM 1 (STRUCTURE OF A NASH EQUILIBRIUM). *A Nash equilibrium always exists in the effort game in the EP model, and is given by the production effort profile (x_i^*, x_{-i}^*) that satisfies,*

$$x_i^* = \left[1 - \frac{\beta}{1+b} \sum_{j\in T_i\setminus\{i\}} h_{ij}p_{ij}(\mathbf{x}^*_{P_{i_-\to j}})x_j^*\right]^+ \qquad (4)$$

Proof. The existence is immediate since the strategy spaces are compact and the utilities are continuous [11]. The focus of the proof is to get a closed form expression of the equilibrium, and is provided in the Appendix. ■

This theorem shows that the EP model allows us to guarantee the existence of (at least one) Nash equilibrium. In particular, we can make certain observations on the equilibrium production effort, some of which are intuitive.

- If communication improves, i.e., β becomes small, the production effort of each node increases.
- If the cost of management b increases, the production effort of each node increases.
- When reward sharing (h_{ij}) is large, agents reduce production effort and focus more on communication effort, which is more productive in terms of payoffs.
- The computation of a Nash equilibrium at any node depends only on the production efforts of the nodes in its subtree. Thus, we can employ a backward induction algorithm which exploits this property that helps in an efficient computation of the equilibrium (this will be shown formally in the corollaries later in this section).

We turn now to establishing conditions for the uniqueness of this Nash equilibrium. Let us define the maximum reward share that any node i can accumulate from a hierarchy T given a reward sharing scheme H as, $h_{\max}(T) = \sup_i \sum_{j\in T_i\setminus\{i\}} h_{ij}$. We also define the *effort update function* as follows.

DEFINITION 1 (EFFORT UPDATE FUNCTION (EUF)). *Let the function $F : [0,1]^n \to [0,1]^n$ be defined as,*

$$F_i(\mathbf{x}) = \left[1 - \frac{\beta}{1+b} \sum_{j\in T_i\setminus\{i\}} h_{ij}p_{ij}(\mathbf{x}_{P_{i_-\to j}})x_j\right]^+ .$$

Note that the RHS of the above expression contains the production efforts of all the agents in the subtree of agent i. This function is a prescription of the choice of the production effort of agent i, if the agents below the hierarchy choose a certain effort profile. Hence the name 'effort update'.

THEOREM 2 (SUFFICIENCY FOR UNIQUENESS). *If $\beta < \sqrt{\frac{1+b}{h_{\max}(T)}}$, the Nash equilibrium effort profile (x_i^*, x_{-i}^*) is unique and is given by Equation (4).*

Proof sketch. The proof idea here is to show that F is a contraction, and x^* is the unique fixed point of F. ■

THEOREM 3 (TIGHTNESS). *The sufficient condition of Theorem 2 is tight.*

Proof. Consider a 3 node hierarchy with nodes 2 and 3 being the children of node 1 (Figure 2). We show that if the sufficient condition is just violated, it results in multiple equilibria. Let $b = 0$, and $h_{12} = h_{13} = 0.25$, therefore $h_{\max}(T) = 0.25$. Theorem 2 requires that $\beta < 1/\sqrt{0.25} = 2$. We choose $\beta = 2$. The equilibrium efforts for node 2 and 3 are 1. Node 1 solves the following equation to find the equilibria.

$$1 - x_1 = e^{-2x_1}.$$

This equation has multiple solutions, $x_1 = 0, 0.797$, showing non-uniqueness. ■

The uniqueness condition indicates that the communication quality needs to be 'good enough' (small β) to ensure uniqueness of an equilibrium. It is worth noting that the uniqueness condition ensures the convergence of the best response dynamics, in which all the players start from any arbitrary effort profile $\mathbf{x}_{\text{init}}$, and sequentially update their efforts via the function F, to the unique equilibrium. This is a consequence of the fact that F is a contraction.

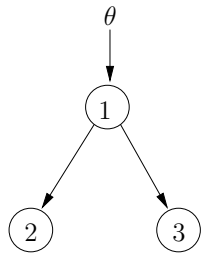

Figure 2: Tightness of the sufficiency (Theorem 2).

We now turn to the computational complexity of a Nash equilibrium. If there are multiple NE, the following corollary holds for computing *a* NE.

COROLLARY 1. *The worst-case complexity of computing the equilibrium effort for node i is $O(|T_i|^2)$. As a result, The worst-case complexity of computing the equilibrium efforts of the whole network is $O(n^2)$.*

Proof. To compute the equilibrium production effort x_i^*, node i needs to compute Equation (4), which requires to compute the equilibrium efforts for each node in i's subtree T_i. Because of the fact that x_i^* depends only on the equilibrium efforts of the subtree below i, we can apply the backward induction method starting from the leaves towards the root of this sub-hierarchy T_i. The worst-case complexity of such a backward induction occurs when the sub-hierarchy is a line. In such a case the complexity would be $|T_i|(|T_i|-1)/2 = O(|T_i|^2)$. In order to compute the equilibrium efforts of the whole network, it is enough to determine the equilibrium effort at the root because this would, in the process, determine the equilibrium efforts of each node in the hierarchy. This is also a consequence of the backward induction method of computing the equilibrium. The worst-case complexity of finding the equilibrium effort at the root is $O(n^2)$ and therefore the worst-case complexity of computing the equilibrium efforts of the whole network is also $O(n^2)$. ■

With the characterization results on the Nash equilibrium efforts, we now move on to the focus of this paper, where we design reward sharing scheme in order to maximize the productive output of the crowdsourcing network.

3. MAXIMIZING THE PRODUCTIVE OUTPUT OF THE NETWORK

In our model, the equilibrium behavior of the agents are tightly coupled with the network structure and the reward sharing scheme as seen from Equation (4). In this section, we look at how the equilibrium behavior given a reward sharing scheme affects the *social output* of the hierarchy T for a given effort vector $\mathbf{x} \in [0,1]^n$, defined as follows.

$$SO(\mathbf{x}, T) = \sum_{i \in N} p_i(\mathbf{x}_{P_{\theta \to i}}) x_i \tag{5}$$

This quantity captures the sum of the output of each individual agents in the network, where the output of each agent is the product of their productivity and production effort. For a given hierarchy T, let us define the optimal effort vector as $\mathbf{x}^{\text{OPT}} \in \arg\max_{\mathbf{x}} SO(\mathbf{x}, T)$. This is the production effort profile across the network that maximizes the total direct output value, considering also the effect of communication effort (induced by lower production effort) on the productivity of other nodes. Ideally the designer would like to achieve this maximal social output for the given hierarchy. However, the strategic choice of the individuals might not lead to this performance of the system as a whole. The question we address in this section is how the Nash equilibrium effort level $\mathbf{x}^*$ performs in comparison to the socially optimal outcome $\mathbf{x}^{\text{OPT}}$.

We will consider cases where the equilibrium is unique, hence, the *price of anarchy* [13] is given by:

$$\text{PoA} = \frac{SO(\mathbf{x}^{\text{OPT}}, T)}{SO(\mathbf{x}^*, T)}. \tag{6}$$

This quantity measures the degree of efficiency of the network. Making PoA equal to unity would be the ideal achievement for the designer. However, that may not always be possible given the parameters of the model. In such a case, we provide a design procedure of the reward sharing scheme that yields the maximum social output.

We note that the equilibrium effort profile $\mathbf{x}^*$ depends on the reward sharing scheme H, while $\mathbf{x}^{\text{OPT}}$ does not. The goal of this section is to understand how one can engineer the H to reduce the PoA (thereby making the social output closer to the optimal). The following theorem shows that if the reward sharing is not properly designed, the PoA can be arbitrarily large. We consider a single-level hierarchy (see Figure 3). To simplify the analysis, we also assume that the function $\mu(C_1) = 1$, irrespective of the number of children of node 1. By symmetry, we consider a single value h, such that $h_{12} = h_{13} = \ldots = h_{1n} = h$. We refer to this model as `FLAT`. We show that PoA can be large when there is bad communication (β large) and no profit sharing ($h = 0$).

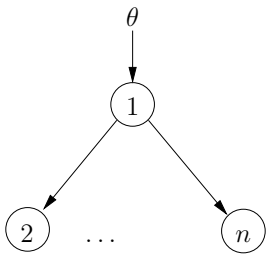

Figure 3: `FLAT` hierarchy.

THEOREM 4 (LARGE POA). *For $n \geq 3$, the PoA is $\frac{n-1}{2}$ in the* `FLAT` *hierarchy when $\beta = \ln(n-1)$ and $h = 0$.*

Proof. For `FLAT`, the social output is given by, $SO(\mathbf{x}, \texttt{FLAT}) = \sum_{i=2}^{n} e^{-\beta x_1} x_i + x_1$. We see that $\beta = \ln(n-1) \geq -\ln\left(1 - \frac{1}{n-1}\right)$, for all $n \geq 3$. It is easy to check that the optimal effort profile that maximizes the social output is $\mathbf{x}^{\text{OPT}} = (0, 1, \ldots, 1)$. Hence the optimal social output is $(n-1)$. However, for reward sharing factor $h = 0$, we get the equilibrium effort profile from Equation (4) to be $\mathbf{x}^* = (1, 1, \ldots, 1)$. This yields a social output of $(n-1)e^{-\beta} + 1$. Hence the PoA is $\frac{n-1}{(n-1)e^{-\ln(n-1)}+1} = \frac{n-1}{2}$. ■

However, if h is chosen appropriately, e.g., if it were chosen to be large positive, the equilibrium effort profile given by Equation (4) would have been closer to that of the optimal. Hence PoA could have been reduced and made closer to 1.

This raises a natural question: *is it always possible to design a suitable reward sharing scheme that can make PoA = 1 for any given hierarchy?* To answer that, we define the *stability* of an effort profile $\mathbf{x}$.

DEFINITION 2 (STABLE EFFORT VECTOR). *An effort profile $\mathbf{x} = (x_1, \ldots, x_n)$ is* stable, *represented by $\mathbf{x} \in S$, if $\mathbf{x} \geq \mathbf{0}$, and there exists a reward sharing matrix $H = [h_{ij}]$, $h_{ij} \geq 0$, such that,*

$$\sum_{j \in T_i \setminus \{i\}} a_{ij}(\mathbf{x}) h_{ij} \geq 1 - x_i; \quad \sum_{j \in T_i \setminus \{i\}} h_{ij} \leq \frac{1+b}{\beta^2}, \ \forall i \in N. \tag{7}$$

Where, $a_{ij}(\mathbf{x}) = \frac{\beta}{1+b} p_{ij}(\mathbf{x}_{P_{i\_ \to j}}) x_j$, for all $j \in T_i \setminus \{i\}$, and zero otherwise. We call the corresponding solution H^* a stable reward sharing matrix.

The inequalities capture a required balance between incentives and information flow. In the first inequality, for

a fixed communication factor β and cost coefficient b, the term $a_{ij}(\cdot)$ is proportional to the fractional output (fractional productivity $\times$ production effort) of an agent j. After multiplying with h_{ij}, this is the effective indirect output that i receives from j. The RHS of the inequality can be interpreted as the communication effort of agent i. Hence, this inequality says that the total indirect benefit should be at least equal to the effort put in by a node for communicating the information to its subtree. If we consider that the agents share information based on the reward share they receive, the flow of information and reward forms a closed loop. The second inequality says that the closed loop 'gain' of the information flow (β^2) and the reward share accumulated by agent i ($\sum_{j\in T_i\setminus\{i\}} h_{ij}$) should be bounded by the cost of sharing the information. The closed loop 'gain' is essentially the reward that an agent accumulates due to his communication effort *through* his descendants. We can connect a stable effort vector with the Nash equilibrium of the effort game.

LEMMA 1 (STABILITY-NASH RELATIONSHIP). *If an effort profile $\mathbf{x} = (x_1, \ldots, x_n)$ is stable, it is the unique Nash equilibrium of the effort game with the corresponding stable reward sharing matrix.*

Proof. Let $\mathbf{x}$ is a stable effort profile. So, there exists a stable reward sharing matrix corresponding to it. Let $H = [h_{ij}],\ h_{ij} \geq 0$ be the matrix, s.t. Equation (7) is satisfied with $\mathbf{x}$. Also $\mathbf{x} \geq \mathbf{0}$. Therefore, reorganizing the first inequality of Equation (7) and noting the fact that $x_i \geq 0,\ \forall i \in N$, we get,

$$x_i = \left[1 - \sum_{j\in T_i\setminus\{i\}} a_{ij}(\mathbf{x})h_{ij}\right]^+,\ \forall i \in N.$$

Under the condition given by the second inequality of Equation (7), the Nash equilibrium is unique and is given by the above expression (recall Theorem 2). Hence, $\mathbf{x}$ is the unique Nash equilibrium of this game. ■

Now it is straightforward to see that the stability of $\mathbf{x}^{\text{OPT}}$ is sufficient for PoA to be 1. This is because now the H that makes the $\mathbf{x}^{\text{OPT}}$ vector stable can be used as the reward sharing scheme, and for that H the equilibrium effort profile will coincide with $\mathbf{x}^{\text{OPT}}$. In other words, the optimal effort vector can be supported in equilibrium by a suitable reward sharing scheme. Hence, the following lemma is immediate.

LEMMA 2 (NO ANARCHY). *A stable reward sharing scheme corresponding to $\mathbf{x}^{OPT}$ yields a PoA of 1.*

A couple of important questions are then: *how efficiently can we check if a given effort profile $\mathbf{x}$ is stable or not? And how to choose a reward sharing scheme that makes the effort profile stable?* The answer is that we can solve the following feasibility linear program (LP) for a given effort profile:

$$\begin{array}{lrcl} \min & & 1 & \\ \text{s.t.} & \sum_{j\in T_i\setminus\{i\}} a_{ij}(\mathbf{x})h_{ij} & \geq & 1 - x_i, \\ & \sum_{j\in T_i\setminus\{i\}} h_{ij} & \leq & \frac{1+b}{\beta^2}, \\ & h_{ij} & \geq & 0, \forall j, \end{array} \Bigg\} \ \forall i \in N. \tag{8}$$

If a solution exists to the above LP, we conclude that $\mathbf{x}$ is stable and declare the corresponding H to be the resulting reward sharing scheme. Linear programs can be efficiently solved and therefore checking an effort profile for stability can be done efficiently.

A Note on the Reward Share Design.

This condition gives us a recipe for reward sharing scheme design. However, the next question is: *what happens when the $\mathbf{x}^{OPT}$ is unstable?* If the above feasibility LP does not return any solution matrix H, we conclude that $\mathbf{x}^{\text{OPT}} \notin S$. In such a scenario, we cannot guarantee PoA to be unity. However, for any given reward sharing matrix H, there is an equilibrium effort profile $\mathbf{x}^*(H)$. We can, therefore, solve for $H_{\max} \in \arg\max_{H:\mathbf{x}^*(H)\in S} SO(\mathbf{x}^*(H))$ which leads to an equilibrium effort profile $\mathbf{x}^*(H_{\max})$ that lies in the stable set and maximize the social output. Therefore, when we cannot find a reward sharing scheme to achieve the optimal social output, $H_{\max}$ is an optimal design of reward share. Computing $H_{\max}$ for general hierarchies may be a hard problem, and we leave that as a future work. However, for certain special classes of hierarchies, it is possible to derive bounds on the PoA (thereby providing a design recipe for H to achieve a lower bound on the social output). In the following section, we do the same for the balanced hierarchies. The price of anarchy analysis, therefore, serves as a means to find the optimal reward sharing scheme that gives a theoretical guarantee on the social output of the system.

3.1 Price of Anarchy in Balanced Hierarchies

While the results in previous sections apply to general hierarchies, we now consider a simple yet representative class of hierarchies, namely the balanced hierarchies, and analyze the effect of communication on PoA and provide efficient bounds. Hierarchies in organizations are often (nearly) balanced, and the `FLAT` or linear networks are special cases of the balanced hierarchy (depth = 1 or degree = 1). Hence, the class of balanced hierarchies can generate useful insights. In addition, the symmetry in balanced hierarchies allows us to obtain interpretable closed-form bounds and understand the relative importance of different parameters.

We consider a balanced d-ary tree of depth D. By symmetry, the efforts of the nodes that are at the same level of the hierarchy are same at both equilibrium and optimality. This happens because of the fact that in the EP model, both the equilibrium and optimal effort profile computation follows a backward induction method starting from the leaves towards the root. Since the nodes in the same level of the hierarchy is symmetric in the backward induction steps, they have identical effort profiles.

With a little abuse of notation, we denote the efforts of each node at level i by x_i. We start numbering the levels from root, hence, there are $D + 1$ levels. Note that there are a few interesting special cases of this model, namely (a) $d = 2$: balanced binary tree, (b) $D = 1$: flat hierarchy, (c) $d = 1$: line. We assume, for notational simplicity only, that the function $\mu(C_k) = 1$, for all C_k, though our results generalize. This function is the coefficient of the productivity function. $\mu(C_k) = 1$ also models organizations where each manager is assigned a small team and there is no attenuation in productivity due to the number of children. In order to present the *price of anarchy* (PoA) results, we define the

set ξ:

$$\xi(\beta) = \left\{ x : x = \left[1 - \frac{1}{\beta} e^{-\beta x} \right]^+ \right\}. \quad (9)$$

This set is the set of possible equilibrium effort levels for agents at the penultimate level of the EP model hierarchy when $\beta > 1$. Note that this set is a singleton, when $\beta > 1$. Depending on β, we define a lower bound $\phi(d, \beta)$ on the contribution of an agent toward the social output, and a sequence of nested functions t_i, where d is the degree of each node.

$$\begin{aligned} \phi(d,\beta) &= \max\left\{ \frac{1}{\beta}(1+\ln(d\beta)), d\beta + (1-d\beta)\xi(\beta) \right\}, \\ t_1(d,\beta) &= \phi(d,\beta), \\ t_2(d,\beta) &= \phi(d \cdot t_1(d,\beta),\beta), \\ &\cdots \\ t_D(d,\beta) &= \phi(d \cdot t_{D-1}(d,\beta),\beta). \end{aligned} \quad (10)$$

THEOREM 5 (PRICE OF ANARCHY). *For a balanced d-ary hierarchy with depth D, as β increases, we can show the following price of anarchy results.*

$$\begin{aligned} &\text{When } 0 \le \beta \le 1, && PoA = 1, \\ &\text{and when } 1 < \beta < \infty, && PoA \le \frac{d^D}{t_D(d,\beta)}. \end{aligned} \quad (11)$$

Proof sketch. The proof is constructive and sets the H matrix appropriately to achieve the bounds on PoA. The H matrix constructed this way acts as the reward sharing scheme to achieve a reasonable enough social output. We skip the complete proof due to space limitations. ■

As opposed to our choice of lower bound ϕ, a naïve lower bound of $\frac{1}{\beta}(1+\ln(d\beta))$ can also be used. The corresponding sequence of nested functions similar to the ones defined in Equation (10) is denoted by $q_i, i = 1, \ldots, D$. However, this gives a weaker bound for any hierarchy. As an example, we demonstrate the weakness for `FLAT` (recall Figure 3) in Figure 4 (the `FLAT` hierarchy is a balanced tree with $D = 1, d = n - 1$). Figure 4 shows that the bound given by our analysis is tight for `FLAT`, indicating the value of the analysis and also gives intuition to the shape of the effect of β on the PoA. We can then have the following corollaries of Theorem 5,

COROLLARY 2 (OPTIMAL EFFORT). *For the `FLAT` hierarchy, if $0 \le \beta < -\ln\left(1 - \frac{1}{n}\right)$, the optimal effort profile is where all nodes put unit effort. When $-\ln\left(1 - \frac{1}{n}\right) \le \beta < \infty$, the optimal changes to the profile where the root node puts zero effort and each other node puts unit effort.*

COROLLARY 3. *For the `FLAT` hierarchy, when $0 \le \beta \le 1$, $PoA = 1$, and when $1 < \beta < \infty$, $PoA \le \frac{n}{\phi(d,\beta)}$.*

The second corollary above makes rigorous the intuition that when β is small enough the optimal $\mathbf{x}$ can be achieved by choosing a small enough reward share h. However, when β grows, in order to ensure uniqueness of the Nash equilibrium, the choice of h becomes limited (as it has to satisfy $\le (1+b)/\beta^2$) resulting in a PoA, as captured in Figure 4.

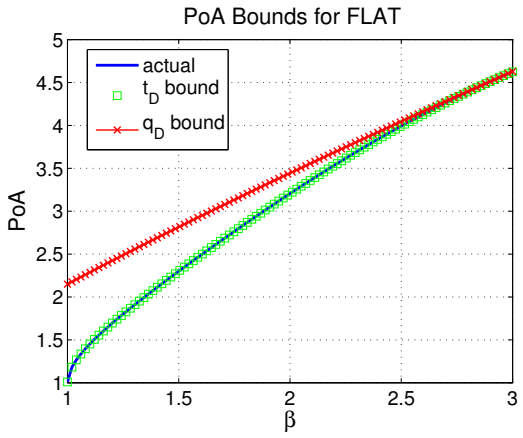

Figure 4: Bounds on PoA for `FLAT`, $d = 6, D = 1$.

4. CONCLUSIONS AND FUTURE WORK

In this paper, we built on the papers [5, 3] and developed an understanding of the effort levels in crowdsourcing hierarchies of influencers and influencees. Taking a game theoretic perspective, we introduce a general utility model, through which we were able to show results on the existence, structure, and uniqueness of Nash equilibrium efforts. For the space limitations, we focused on hierarchical networks, and with the EP model we found closed form expressions and a design recipe for the reward sharing scheme that maximize the productive output of the hierarchy. We show that for a strategic crowd, achieving an optimal productive output may not be possible, and we provided bounds on this achievability via PoA analysis on balanced hierarchies. Our results on existence and uniqueness extend to general directed networks. Finding the output maximizing reward sharing scheme design for non-hierarchical networks stands as an interesting future work.

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APPENDIX

Proof of Theorem 1

Proof. Given that the existence is a corollary of [11], we are left to show that a Nash equilibrium profile (x_i^*, x_{-i}^*) must satisfy Equation (4). For notational convenience, we drop the arguments of p_i and p_{ij}, which are functions of $\mathbf{x}_{P_{\theta\to i}}$ and $\mathbf{x}_{P_{i\_\to j}}$ respectively. Each agent $i \in N$ solves the following optimization problem.

$$\begin{array}{cc} \max_{x_i} & u_i(x_i, x_{-i}) \\ \text{s.t.} & x_i \geq 0 \end{array} \tag{12}$$

Combining Equations (1), (2), and (3), we get,

$$u_i(x_i, x_{-i}) = p_i(\mathbf{x}_{P_{\theta\to i}}) \left(x_i - \frac{x_i^2}{2} - b\frac{(1-x_i)^2}{2} \right) + \sum_{j \in T_i \setminus \{i\}} h_{ij} p_j(\mathbf{x}_{P_{\theta\to j}}) x_j.$$

Note that we have relaxed the constraint from $0 \leq x_i \leq 1$. The first additive term in the utility function has the peak at $x_i = 1$. The second term has $e^{\beta x_i}$ in the p_j, which is decreasing in x_i. Therefore, the optimal x_i that maximizes this utility will be ≤ 1. Hence, in this problem setting, the optimal solution for both the exact and the relaxed problems is the same. So, it is enough to consider the above problem. For this non-linear optimization problem, we can write down the Lagrangian as follows.

$$\mathcal{L} = u_i(x_i, x_{-i}) + \lambda_i x_i, \ \lambda_i \geq 0.$$

The KKT conditions for this optimization problem (12) are:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, \Rightarrow \frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) + \lambda_i = 0, \tag{13}$$

$$\lambda_i x_i = 0, \text{ complementary slackness.} \tag{14}$$

Case 1: $\lambda_i = 0$, then from Equation (13) we get,

$$p_i(1 - x_i + b(1 - x_i)) + \sum_{j \in T_i \setminus \{i\}} h_{ij} \frac{\partial p_j}{\partial x_i} x_j = 0$$

$$\Rightarrow \quad p_i(1+b)(1-x_i) - \beta \sum_{j \in T_i \setminus \{i\}} h_{ij} p_j x_j = 0$$

$$\Rightarrow \quad 1 - x_i = \frac{\beta}{1+b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j, \text{ with } p_{ij} \text{ as defined}$$

$$\Rightarrow \quad x_i = 1 - \frac{\beta}{1+b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j. \tag{15}$$

Case 2: $\lambda_i > 0$, then from Equation (14) we get $x_i = 0$, and from Equation (13), $\frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) < 0$. Carrying out the differentiation as in Equation (15) we get,

$$0 = x_i > 1 - \frac{\beta}{1+b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j. \tag{16}$$

$$\therefore \ x_i = \left[1 - \frac{\beta}{1+b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j \right]^+.$$

Since this condition has to hold for all nodes $i \in N$, the equilibrium profile (x_i^*, x_{-i}^*) must satisfy the above equality. ■