Optimal Referral Auction Design

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Abstract

The auction of a single indivisible item is one of the most celebrated problems in mechanism design with transfers. Despite its simplicity, it provides arguably the cleanest and most insightful results in the literature. When the information that *the auction is running* is available to every participant, Myerson [22] provided a seminal result to characterize the incentive-compatible auctions along with revenue optimality. However, such a result does not hold in an *auction on a network*, where the information of the auction is spread via the agents, and they need incentives to forward the information. In recent times, a few auctions (e.g., [15, 20]) were designed that appropriately incentivized the intermediate nodes on the network to *promulgate* the information to potentially more valuable bidders. In this paper, we provide a Myerson-like characterization of *incentive-compatible auctions on a network* and show that the currently known auctions fall within this class of randomized auctions. We then consider a special class called the *referral auctions* that are inspired by the multi-level marketing mechanisms [1, 7, 8] and obtain the structure of a revenue optimal referral auction for i.i.d. bidders. Through experiments, we show that even for non-i.i.d. bidders there exist auctions following this characterization that can provide a higher revenue than the currently known auctions on networks.

1 Introduction

Single indivisible item auction is a special setting of mechanism design with monetary transfers where multiple bidders contest to collect a single item. The *true value* of the item could be different for different agents and it is their *private information*, i.e., not known to the designer of a *mechanism*.[1] Despite its simplicity, single-item auction provides remarkable insights into the following questions: (a) what is the structure of the mechanisms that reveal the agents' true private information, (b) how to design mechanisms that maximize the expected revenue. In a world where the information that 'an item is being auctioned' is available to every possible bidder interested in this item, these two questions have been answered gracefully by Myerson in his seminal paper [22].

However, in various recent contexts of auctions, the network of connections makes an important role in the information flow over the network. An agent diffuses the information into the network only if it finds it is *beneficial* to share. This setup is called *network auctions*, where agents *diffuse* the information of the auction only if it (strictly or weakly) improves their utilities. This problem has given birth to the domain of *diffusion auction design* on networks and has received significant attention in the recent times [11, 15, 19, 20]. Because the information about the auction does not *automatically* reach every agent in this setup, the mechanism needs to incentivize the individuals to diffuse (or forward) the information. The Myerson [22] characterization does not follow here,

[1]Since auctions are special cases of mechanisms, we will use these two terms interchangeably in this paper.

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and a fresh investigation is necessary to characterize the *truthful* and *revenue maximizing* diffusion auctions.

1.1 Our contributions

The contributions of this paper are divided into two parts. In the first part, we characterize the *truthful* network auction via certain constraints on allocation and payment. In the second part, we consider a Bayesian setup and find the class of revenue-optimal network auctions. More concretely:

1. We provide a more direct definition of truthfulness called *diffusion dominant strategy incentive compatibility* (`DDSIC`, Def. 2) and show that it is equivalent to the existing IC definition in the literature on network auction [19, e.g.].
2. We characterize `DDSIC` (and therefore, IC) mechanisms (Theorems 1 to 3 and corollary 1) with constraints on the allocation and payment of the auctions. Note that this result is of independent interest irrespective of the revenue optimality question addressed later in this paper (similar to [22]).
3. We introduce a new mechanism called `LbLEV` that is `DDSIC` (Theorem 4) and individually rational (Theorem 5) on a tree.
4. We find the revenue-optimal *referral auction* (which is a class of auctions motivated by the multi-level marketing methods) for i.i.d. bidders (Theorem 9) in §6.
5. When bidders are non-i.i.d., we experimentally exhibit that the parameters of `LbLEV` can be tuned, based on the prior information of the valuations and network structure, to yield a better revenue than the currently known truthful diffusion auctions (§7).

1.2 Related work

The area of diffusion auction design is relatively new, leading to a rather thin literature. Guo and Hao [11] provide a comprehensive survey of the domain. The first works on diffusion auction are due to Li et al. [15] and Lee [14]. In particular, [15] showed that the classical VCG mechanism [4, 10, 28] can be extended to the diffusion setting, but it may lead to a large deficit. They propose a new mechanism called IDM that mitigates this problem. In the following years, a few more diffusion auctions were proposed: CSM [16] for economic networks, MLDM for intermediary networks [18], TNM, CDM, WDM were on the unweighted and weighted networks [17, 20], FDM [29] and NRM [30] considered the money burning issue in network auction and proposed schemes to redistribute the money maintaining incentive compatibility. On the characterization results, Li et al. [19] provide a characterization for deterministic diffusion auctions and find optimal payments. Our approach, however, considers a broader approach to characterize all *randomized* diffusion auctions (which includes the deterministic auctions as a special case) and shows that better revenue-generating mechanisms can be found.

On the other hand, auctions are fairly well understood in the setting without networks, both in theory [5, 22, 24] and in practice [13]. The primary focus of the paper is to extend the theory to network auctions. We begin by understanding revenue optimality in a simpler class of mechanisms, namely the *referral auctions*, which has a close similarity with a business method called *multi-level marketing* (MLM) (a survey can be found in [25]). Direct sales firms often use this method to encourage individual distributors to recruit new distributors. It is a multi-billion dollar industry (US figures available in [23]) – according to a 2018 survey, 7.7% of the US adult population had participated in at least one MLM business during their lifetime [6]. A prominent example of an MLM scheme is the DARPA red balloon challenge.[2] MLM mechanisms are also well investigated in the mechanism design literature [1, 7, 8]. Despite their criticism for being similar to pyramid schemes, Nat and Keep [23] show some important differences that keep MLM mechanisms relevant in practice. A diffusion auction incentivizes agents on a network to refer to more individuals who are potentially interested in participating in the auction. Hence, in this paper, we consider a natural candidate class of *referral auctions* for revenue optimality (§ 6.2).

2 Basic Problem Setup

Consider a directed graph $G = (N \cup \{s\}, E)$, where $N = \{1, \ldots, n\}$ is the set of players involved in the auction of a single indivisible item and s is a distinguished node called the *seller*. The set E is

[2] `https://www.darpa.mil/about-us/timeline/network-challenge`

the set of edges. Each edge (i, j) denotes that if node i shares information, and node j will receive it. Typical examples of such graphs are online social networks where an individual can share information (selectively) with a subset of her neighbors. The direction signifies that almost all networks have asymmetric information flow (e.g., only followers receive the information from the followee).

In this network, node s is a single seller that wants to sell the indivisible item. Every other node $i \in N$ is a potential buyer and the information about the auction flows only via the direction of an edge. The information cannot reach a node unless there is a directed path from s to that node and each intermediate node decides to forward the information. An intermediate node may decide not to forward the information if it reduces its *utility*.

This setup naturally brings up an auction-like information-sharing game among the players. Each player $i \in N$ has a type $\theta_i = (v_i, r_i)$, where v_i is the valuation of agent i for the item, and r_i is the set of her directed neighbors. The set of valuations and subsets of neighbors of player i are denoted by $\mathcal{V}_i$ and $\mathcal{R}_i$ respectively. The type set of i, Θ_i, is therefore, $\mathcal{V}_i \times \mathcal{R}_i$, and agent i can report its type from this set. The information about the auction needs to reach via directed edges to player i for her to participate in the auction. Therefore, the auction asks every agent to report her valuation for the item and to forward the information to its directed neighbors. In our model, this is captured via their reported type $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i)$ for every agent $i \in N$. We assume that the seller s is not a strategic player in this auction, rather he wants to sell the object and always forwards the information to his directed neighbors. The vector of the reported types of all the agents except i is denoted by $\hat{\theta}_{-i} = (\hat{\theta}_1, \ldots, \hat{\theta}_{i-1}, \hat{\theta}_{i+1}, \ldots, \hat{\theta}_n)$. We denote the set of all type profiles by $\Theta := \prod_{i \in N} \Theta_i$.

Depending on the reported types of the agents, particularly, the reported $\hat{r}_i$'s, the auction may reach only a subset of the agents in N. Throughout this paper, we will use the notation r_i to denote the *true* neighbor set of player i and assume that the reported $\hat{r}_i$ can only be a subset of it. To denote the reported valuation and directed neighbors on the subnetwork generated by $(\hat{\theta}_i, \hat{\theta}_{-i})$, we use a *filter function* f^G for the graph G, where $f^G(\hat{\theta}_i, \hat{\theta}_{-i})$ denotes the reported valuation and directed neighbor vector of the subgraph where each node has a directed path from s after the agents reported the type profile $(\hat{\theta}_i, \hat{\theta}_{-i})$. In this setup, the auction design goal is to incentivize each node to truthfully reveal its private valuation and forward regardless of others' actions. It is known that such mechanism exists [15]. One of the goals of this paper is to characterize all such mechanisms in an elegant manner.

We consider auctions on this graph with randomized allocations to the agents. Formally, we define a *diffusion auction* in this setup as follows.

Definition 1 (Diffusion Auction) *A* diffusion auction (DA) *is given by the tuple* (g, p) *where* g *and* p *are the* allocation *and* payment *functions respectively. The allocation function* $g : \Theta \to \Delta_n$ *is such that its* i*-th component* $g_i(f^G(\theta))$ *denotes the probability of agent* i *winning the object, where* $\Delta_n := \{x \in \mathbb{R}^n_{\geqslant 0} : \sum_{i=1}^n x_i = 1\}$*. Similarly, the payment function* $p = (p_i)_{i \in N}$ *is such that its* i*-th component* $p_i : \Theta \to \mathbb{R}$ *denotes the payment assigned to agent* i*.*

Note that g_i should operate on the subnetwork that remains connected to s after the agents choose their actions $\hat{\theta}$. Hence the notation $g_i(f^G(\cdot))$ is used in the definition above. It is worth noting that the notation generalizes the one used by Li et al. [15]. The action chosen by player i may change the actions available to the other players and it is succinctly captured by the filter function which also subsumes the definition in that paper. Also, note that DA is different from the classical auction, because the types of each agent now contain both the valuation (v_i) and the information on forwarding (r_i). The utility of agent i under DA is given by the standard quasi-linear model [27]: $u_i^{(g,p)}((\hat{\theta}_i, \hat{\theta}_{-i}); \theta_i) = v_i g_i(f^G(\hat{\theta}_i, \hat{\theta}_{-i})) - p_i(f^G(\hat{\theta}_i, \hat{\theta}_{-i}))$.

3 Design Desiderata

The first desirable property of an auction is truthfulness. However, in the context of auctions on the network, we need to ensure that the mechanism also incentivizes the agents to forward the information in addition to being truthful about their valuations. The following definition captures both these aspects.

Definition 2 (Diffusion Dominant Strategy Incentive Compatibility) *A* DA (g, p) *on a graph* G *is* diffusion dominant strategy incentive compatible (DDSIC) *if*

1. *every agent's utility is maximized by reporting her true valuation irrespective of the diffusing status of herself and the other agents, i.e., for every $i \in N$, $\forall r_i, \hat{\theta}_{-i}$, the following holds*

$$\begin{aligned} &v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) \\ &\geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})), \forall v_i, v_i', \hat{\theta}_{-i}, r_i' \subseteq r_i, \text{ and,} \end{aligned}$$

2. *for every true valuation, every agent's utility is maximized by diffusing to all its neighbors irrespective of the diffusion status of the other agents, i.e., for every $i \in N$, $\forall r_i, \hat{\theta}_{-i}$, the following holds*

$$\begin{aligned} &v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \\ &\geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})), \forall v_i, \hat{\theta}_{-i}, r_i' \subseteq r_i. \end{aligned}$$

We show later in this paper that the above definition is equivalent to the following definition of incentive compatibility (restated below with the notation of this paper) given by Li et al. [19] and hence can be used interchangeably.

Definition 3 (Incentive Compatibility [19]) *A* DA (g, p) *on a graph G is incentive-compatible (IC) if for every $i \in N$, $\forall r_i, \hat{\theta}_{-i}$, $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})), \forall v_i, v_i', \forall r_i' \subseteq r_i, \forall i \in N$.*

Why DDSIC? A natural question can arise: why do we introduce a new definition of truthfulness when there is an existing one, given that both are equivalent? This is because the new definition provides a more direct and intuitive way to understand the truthful reporting of valuation and diffusion. DDSIC does this by splitting the IC condition into two sets of inequalities as given in Def. 2. In our proofs, this definition makes the analysis of truthful mechanisms simpler. We will show that IC and DDSIC are equivalent and both are equivalent to the two conditions stated in Theorem 1, and will subject all our further analyses only to DDSIC.

The next desirable property deals with the participation guarantee of the agents.

Definition 4 (Individual Rationality) *A* DA (g, p) *on a graph G is* individually rational (IR) *if $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant 0$, $\forall v_i, r_i, \hat{\theta}_{-i}, \forall i \in N$.*

4 Characterization Results

Our first result is to characterize the IC diffusion auctions and show equivalence between IC and DDSIC. The result by Myerson [22] in the single indivisible item auction setup implicitly assumes that *the knowledge of auction reaches all the players for free*, and therefore, no additional incentive is required for the agents to *diffuse* the information of the auction into the network. But in our setting, the information reaches an agent on a network only if *every* predecessor in *at least* one *path* from the seller to that agent forwards this information. Our result, therefore, generalizes Myerson's characterization result in network auctions. For a cleaner presentation, we define the following class of payments.

Definition 5 (Monotone and Forwarding-Friendliness (MFF)) *For a given network G, a* DA (g, p) *is* monotone and forwarding-friendly (MFF) *if*

(a) *the functions $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ are monotone non-decreasing in v_i, for all $r_i, \hat{\theta}_{-i}$, and $i \in N$, and for the given allocation function g, the payment p_i for each player $i \in N$ is such that, for every v_i, r_i, and $\hat{\theta}_{-i}$, the following two conditions hold.*

(b) *For every $r_i' \subseteq r_i$, the following payment formula is satisfied.*

$$p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) = p_i(f^G((0, r_i'), \hat{\theta}_{-i})) + v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - \int_0^{v_i} g_i(f^G((y, r_i'), \hat{\theta}_{-i}))\, \mathrm{d}y \tag{1}$$

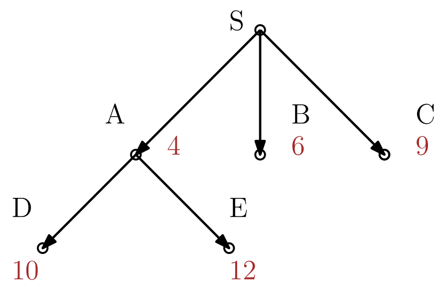

Figure 1: A randomized DDSIC auction.

(c) For every $r'_i \subseteq r_i$, the values of $p_i(f^G((0, r'_i), \hat{\theta}_{-i}))$ and $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$ satisfies the following inequality.

$$p_i(f^G((0, r'_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) \geqslant \int_0^{v_i} \Big(g_i(f^G((y, r'_i), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \Big) \, \mathrm{d}y. \tag{2}$$

We will refer to $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$, the first term on the RHS of Eqn. (1), as the *value independent payment component* (VIPC) in the rest of the paper, since this component of player i is not dependent on the valuation of i.

Theorem 1 (IC $\Rightarrow$ MFF) *If a DA (g, p) is IC, then it is MFF.*

Discussions The characterization result is much in the spirit of the result of Myerson [22]. However, there are the following important observations on these results.

- The payment component of MFF (Def. 5) has two conditions given by Eqns. (1) and (2). While Eqn. (1) is reminiscent of Myerson [22], the important difference here is in the VIPC terms. These terms in the payment formula of Myerson [22] were unrestricted for the characterization of DSIC. However, for DDSIC, Eqn. (2) puts additional constraints on the VIPCs.
- Our result is unique since we provide a characterization of all *randomized* single indivisible item auctions. The closest characterization result to our knowledge applies to only deterministic auctions [19]. The example of the following DDSIC auction is not covered by the characterization of [19] but is covered under Theorem 1. We also provide a class of mechanisms later in this paper which subsumes many currently known mechanisms that are DDSIC.

4.1 Example to illustrate the conditions of a randomized DDSIC auction.

The distinguishing factor of the truthfulness guarantee given by DDSIC is in the part where an agent may not diffuse the information to its neighbors. In this example, we will focus only on that part and illustrate the meaning of the conditions of MFF (Def. 5), that is equivalent to DDSIC via Corollary 1. This example can be easily extended to a full-fledged randomized DDSIC auction. However, that needs the auction to be defined for *every* realized graph and for *every* type profile $((v_i, r_i), \hat{\theta}_{-i})$, which will digress a reader from the main intuition of MFF. Instead, we have explained how these conditions are met when the agents report their (v_i, r_i)s as shown in Fig. 1. For simplicity of exposition, we consider the auction where the true underlying network and the reported valuations are given by Fig. 1, and r_i can take values only in $\{0, 1\}$, i.e., either forward to all its neighbors or not forwarding at all. We discuss the satisfaction of the MFF conditions and consider the variation of v_i and r_i of each agent i keeping the $\hat{\theta}_{-i}$ fixed at the values given in this figure. In this example, agents D and E get the information of the auction only if A forwards it at the first level of the tree. A DDSIC DA (g, p) needs to decide the allocations $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$, and the VIPC components $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$ for $r_i = 0, 1$ and for all $i \in N$. For all agents $i \neq A$, r_i's do not matter since they do not have any children in this tree. Therefore, g_i's and VIPC$_i$'s of agents $i \neq A$, remain unchanged in this example auction when they set $r_i = 0$ or $r_i = 1$ given other agents' reported types are fixed. Hence, condition 2 of MFF is trivially satisfied for all agents except A. We discuss agent A's satisfiability of condition 2 separately later. In a nutshell, this example mechanism adapts the *residual claimant* (RC) mechanism by Green and Laffont [9] to this setting at the first level of the tree. If agent A forwards, then it divides A's probability of allocation with its children and adjusts the payments according to

Def. 5. If A does not forward, then it is just RC. Based on the forwarding decision of A, this example can be divided into two cases:

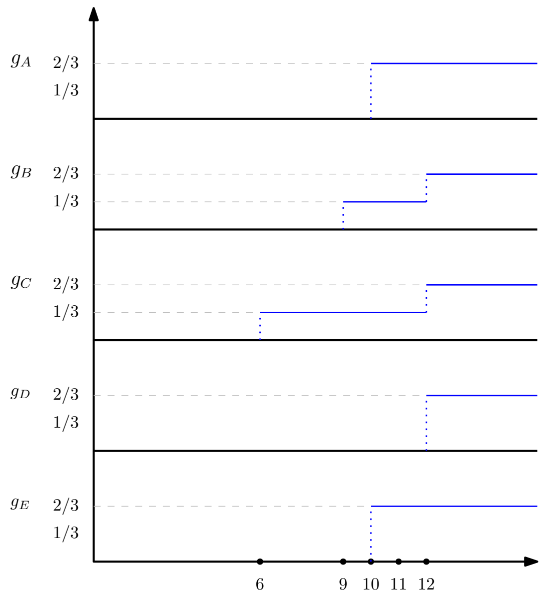

Figure 2: Allocation functions of the nodes in Fig. 1.

Case 1: $r_A = 0$: When agent A does not forward the information, the auction stays limited to the agents A, B, and C. Let the auction give the object w.p. $2/3$ to the highest bidder and w.p. $1/3$ to the second highest bidder. The payment of the highest bidder is $\frac{1}{3}\times$ the second highest bid. This payment is equally distributed among the non-winning agents, which, in this case, is the third highest bidder. This is the modified version of the *residual claimant* mechanism [9], which is DSIC (equivalent to DDSIC for a single-level tree). Hence, under this case, the allocation probability of each agent is clearly monotone non-decreasing since it increases from zero to $1/3$ when it becomes the second highest bidder and from $1/3$ to $2/3$ when it becomes the highest bidder. The VIPC for each agent i is given by $-\frac{1}{3}\times$ the second highest bid in the population except agent i. Therefore, $\texttt{VIPC}_A(r_A = 0) = -6/3 = -2, \texttt{VIPC}_B(r_A = 0) = -4/3, \texttt{VIPC}_C(r_A = 0) = -4/3$. The payments follow from condition 1: $p_A(r_A = 0) = -2 + 0 + 0, p_B(r_A = 0) = -4/3 + 6 \times \frac{1}{3} - \frac{1}{3}(6-4) = 0, p_C(r_A = 0) = -4/3 + 9 \times \frac{2}{3} - \frac{1}{3}(6-4) - \frac{2}{3}(9-6) = 2$. The allocation probabilities are zero for every valuation of agents D and E, and their VIPCs are zeros. Consequently, their payments are also zero in this case.

Case 2: $r_A = 1$: Let the g_i's be given by Fig. 2, when agents except i report their valuations as shown in Fig. 1. Clearly, these are monotone non-decreasing. Let $\texttt{VIPC}_A(r_A = 1) = -11/3, \texttt{VIPC}_B(r_A = 1) = -3, \texttt{VIPC}_C(r_A = 1) = -2, \texttt{VIPC}_D(r_A = 1) = 0, \texttt{VIPC}_E(r_A = 1) = 0$. The payments are given by condition 1 as follows: $p_A(r_A = 1) = -11/3 + 0 + 0, p_B(r_A = 1) = -3 + 0 + 0, p_C(r_A = 1) = -2 + 9 \times \frac{1}{3} - \frac{1}{3}(9-6) = 0, p_D(r_A = 1) = 0 + 0 + 0, p_E(r_A = 1) = 0 + 12 \times \frac{2}{3} - \frac{2}{3}(12-10) = 20/3$.

The satisfiability of condition 2 for agent A warrants a separate discussion since it is the only agent which has a different VIPC when $r_A = 0$ and $r_A = 1$, keeping the other agents' reported types fixed. For all other agents, the satisfiability of condition 2 is trivial since both sides of the inequality reduces to zero. However, we note that the LHS of condition 2 for A is $\texttt{VIPC}_A(r_A = 0) - \texttt{VIPC}_A(r_A = 1) = -2 + 11/3 = 5/3$ which is larger than the RHS for every value of v_A. In particular, when $v_A \geqslant 10$, the RHS becomes $5/3$ and stays constant at that value for larger values of v_A. Hence, condition 2 is satisfied for agent A too.

This is a randomized DA that satisfies the MFF conditions (Def. 5) for $\hat{\theta}_{-i}$ given by Fig. 1. For the $r_A = 1$ case, we could have chosen any monotone allocation rule for the agents and decided the payments according to condition 1, and set arbitrary VIPC terms for agents except A. But for A, we need to ensure that the differences in the VIPCs between $r_A = 0$ and $r_A = 1$ satisfies condition 2. This is the recipe for extending this example for every $\hat{\theta}_{-i}$.

Proof: [of Theorem 1] We begin with an IC mechanism (g, p), and show that the conditions of the theorem hold for this mechanism.

IC implies condition (a): the monotonicity of allocation functions. By definition, IC holds for all v_i', and $r_i' \subseteq r_i$. Therefore, in particular, it must hold for $r_i' = r_i$. We get the following inequality:

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i}) \geq v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}) \quad (3)$$

Adding and subtracting $v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i})$ on the RHS of Eqn. (3), we get:

$$\begin{aligned}
& v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \\
& \geq v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})) \\
\implies & u_i((v_i, r_i), \hat{\theta}_{-i}) \geq u_i((v_i', r_i), \hat{\theta}_{-i}) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))
\end{aligned}$$

From convex analysis [26], we know that the above inequality implies that $g_i(f^G((v_i', r_i), \hat{\theta}_{-i})$ is a sub-gradient of u_i at v_i', if u_i can be shown to be convex in v_i for every $r_i, \hat{\theta}_{-i}$, for every $i \in N$. In the following, we show that it is indeed true.

For brevity, we use the shorthand $h(v_i) := u_i((v_i, r_i), \hat{\theta}_{-i})$ and $\phi(v_i') := g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))$. Because v_i and v_i' were arbitrary in the above inequality, we can choose arbitrary $x_i, z_i \in \mathcal{V}_i$ and define $y_i = \lambda x_i + (1 - \lambda) z_i$ where $\lambda \in [0, 1]$. From the above inequality, we get

$$h(x_i) \geqslant h(y_i) + \phi(y_i)(x_i - y_i) \quad (4)$$
$$h(z_i) \geqslant h(y_i) + \phi(y_i)(z_i - y_i). \quad (5)$$

Multiplying Eqn. (4) by λ and Eqn. (5) by $(1-\lambda)$ and adding, we get $\lambda h(x_i) + (1-\lambda) h(z_i) \geqslant h(y_i)$, which proves that h or the utility u_i is convex, and ϕ or the allocation g_i is its sub-gradient. Since sub-gradient of a convex function is non-decreasing, we get the claimed implication.

IC implies conditions (c) and (b): the payment formulae. Again from convex analysis, we know that for any convex function h having subgradient ϕ, the following integral relation holds: $h(y) = h(z) + \int_z^y \phi(t) dt$ for any y, z in the domain of h. Therefore, using the same definitions of h and ϕ from the previous case, we get (when i diffuses to r_i)

$$\begin{aligned}
& u_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = u_i(f^G((0, r_i), \hat{\theta}_{-i})) + \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i})) dt \\
\Rightarrow\ & v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = -p_i(f^G((0, r_i), \hat{\theta}_{-i})) + \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i})) dt \\
\Rightarrow\ & p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = p_i(f^G((0, r_i), \hat{\theta}_{-i})) + v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i})) dt
\end{aligned}$$

This is precisely Eqn. (1), which is condition (b) of MFF (Def. 5). To prove condition (c) of Def. 5, we first put $v_i' = v_i$ in the definition of IC to get point 2 of Def. 2 and we substitute the payment expressions derived above to get

$$\begin{aligned}
& \int_0^{v_i} g_i(f^G((y, r_i), \hat{\theta}_{-i})) dy - p_i(f((0, r_i), \hat{\theta}_{-i})) \geqslant \int_0^{v_i} g_i(f^G((y, r_i'), \hat{\theta}_{-i})) dy - p_i(f((0, r_i'), \hat{\theta}_{-i})) \\
\Rightarrow \quad & p_i(f((0, r_i'), \hat{\theta}_{-i})) - p_i(f((0, r_i), \hat{\theta}_{-i})) \geqslant \int_0^{v_i} \left(g_i(f^G((y, r_i'), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \right) dy.
\end{aligned}$$

The first inequality follows directly where the terms $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ cancels out on the LHS and $v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$ cancels out on the RHS. The second inequality follows by rearranging the first. Hence, this proves Eqn. (2), condition (c) of Def. 5. Hence, conditions (a), (c), and (b) hold for a mechanism that is Incentive Compatible. ■

We now show that `MFF` implies `DDSIC` .

Theorem 2 (`MFF` $\Rightarrow$ `DDSIC`) *If a* `DA` (g, p) *is* `MFF`*, then it is* `DDSIC`.

Proof:

Conditions (a) and (b) of Def. 5 $\Rightarrow$ point 1 of DDSIC. We are given that $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ is monotone non-decreasing in v_i, for a diffusion type r_i (i.e. when agent i diffuses to neighbour set r_i) and payment is given by Eqn. (1), for all $i \in N$. Assuming v_i to be the true valuation of agent i, the utility of agent i when she is truthful is given by

$$
\begin{aligned}
&v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \qquad (6)\\
&= v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) - v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) + \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt,
\end{aligned}
$$

and the utility when she misreports to v_i' is given by

$$
\begin{aligned}
&v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})) \qquad (7)\\
&= v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) - v_i' g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) + \int_0^{v_i'} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt.
\end{aligned}
$$

Subtracting Eqn. (7) from Eqn. (6), we get

$$
\begin{aligned}
&v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - [v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}))]\\
&\quad = (v_i' - v_i) g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) + \int_{v_i'}^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt. \qquad (8)
\end{aligned}
$$

Since g_i is monotone non-decreasing and non-negative, the RHS of Eqn. (8) is always non-negative. Hence, we have point 1 of DDSIC

Condition (c) $\Rightarrow$ point 2 of DDSIC. Here we have Eqn. (2) satisfied. Given that we also have the expression of the payment given by Eqn. (1) satisfied, adding and subtracting $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ on the LHS of Eqn. (2) and adding and subtracting $v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$ on the RHS and then rearranging, we get point 2 of DDSIC. ■

Our final result in this section is that our definition of truthfulness, DDSIC, implies IC.

Theorem 3 (DDSIC $\Rightarrow$ **IC**) *If a* DA (g, p) *is* DDSIC*, then it is IC.*

Proof: In this proof, we will exhaustively list all the cases of manipulation under Def. 3 and show that each of the inequalities is implied by the conditions of DDSIC (Def. 2). Suppose, (v_i, r_i) is the tuple of the *true* valuation and neighbor set of agent i. The following cases of manipulation in v_i and r_i are exhaustive for Def. 3.

- Case 1: (v_i, r_i'), i.e., valuation is truthfully reported but diffusion is strategized. So, for $v_i' = v_i$ $r_i' \subseteq r_i$, the inequality of Def. 3 becomes

  $$
  v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) \quad \forall v_i, \hat{\theta}_{-i}, \forall i \in N.
  $$

  This is implied by condition 2 of Def. 2.
- Case 2: (v_i', r_i), i.e., valuation is manipulated but diffusion is not. Hence, with $r_i' = r_i$, Def. 3 becomes

  $$
  v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})) \quad \forall v_i, v_i', \hat{\theta}_{-i}, \forall i \in N.
  $$

  This is implied by condition 1 of Def. 2.
- Case 3: (v_i', r_i'), i.e., both valuation and diffusion are strategized. Then the condition of Def. 3 becomes

  $$
  v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', 0), \hat{\theta}_{-i})) - p_i(f^G((v_i', 0), \hat{\theta}_{-i})), \quad \forall v_i, v_i', \hat{\theta}_{-i}, \forall i \in N. \qquad (9)
  $$

  Now from condition 2 of Def. 2, we get

  $$
  v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))
  $$

$$\forall v_i, \hat{\theta}_{-i}, \forall i \in N,$$

and from condition 1 of Def. 2, we get

$$v_i g_i(f^G((v_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r'_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})),$$
$$\forall v_i, v'_i, \hat{\theta}_{-i}, \forall i \in N.$$

Combining these two, we have

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})) - p_i(f^G((v'_i, r'_i), \hat{\theta}_{-i})),$$
$$\forall v_i, v'_i, \hat{\theta}_{-i}, \forall i \in N,$$

which is Eqn. (9).

This completes the proof. ■

Consolidating the results of Theorems 1 to 3, we get the following corollary.

Corollary 1 DDSIC $\iff$ *IC* $\iff$ MFF.

At the end of Section 3, we discussed why DDSIC is introduced despite its equivalence with IC.

Naturally, known network auctions like IDM [15], TNM [20], etc., are also DDSIC. In the following section, we present an unusual class of DDSIC auctions to demonstrate how it differs from the classical single-item auction without forwarding constraints.

5 A non-trivial DDSIC mechanism

We begin with a novel auction on a tree to show that there exists mechanisms which are not explored yet in the network auction parlance. We call this auction *Level-by-Level Exponential Valuation* (LbLEV) auction. This auction assigns some exponents to the agents and proceeds level-by-level from the root to the leaves.

Level-by-Level Exponential Valuation (LbLEV) mechanism

We first present the high-level idea of our mechanism before formalizing it in Alg. 1. As the name suggests, the mechanism is run at every level of the tree $\hat{T}$ rooted at s (induced by the agent reports) from the root towards the leaves. The mechanism LbLEV is parametrized by a vector $t \in \mathbb{R}^n_{>0}$, where t_i denotes the *exponent* of agent i. Different choices of t and $\hat{T}$ for the same input instance create a class of mechanisms, and we call each of them LbLEV. Let $\hat{T}_i$ be the subtree rooted at node i including i. At each level of this tree, the mechanism sets an *offset* for the parent node(s) of that level, and finds the maximum valuation $v_i^{\max}$ in $\hat{T}_i$, for every i at that level. It deducts the offset from $v_i^{\max}$ to calculate i's *effective valuation* ρ_i and decides which $\hat{T}_i$'s "stay in the game". For the i's that stay, the mechanism considers the largest $\rho_i^{t_i}$ and tentatively sets it as the "winning subtree" and calculates its "actual payment". The mechanism repeats at every next level treating the current root of the tentatively winning subtree as the parent with an updated offset. Alg. 1 details out the description of this mechanism in an algorithmic manner.

In the sub-tree $\hat{T}$, every agent receives the difference between the payment that their children in $\hat{T}$ give to that agent and the payment she makes to her parent. The algorithm terminates either at some leaf node or at a node that has large enough offset such that none of its children 'stay in the game'. We call that node the winner of LbLEV.

Illustration of LbLEV through an example. Suppose the sub-tree $\hat{T}$ generated from the reported graph $\hat{G}$ is as shown in Fig. 3 for 11 nodes (named $A, B, \ldots, K$ in the figure) in addition to the seller S. The tuple next to agent i denotes (v_i, t_i), for all $i = A, B, \ldots, K$, where v_i is the reported valuation and t_i is the exponent set by the mechanism. At `level` = 1 of $\hat{T}$, from Alg. 1 we get `offset`= 0. We find the `effective valuations` to be $\rho_A = 750$, $\rho_B = 6$, $\rho_C = 9$. Now, we observe that ${\rho_A}^{t_A} = 750$ is the highest among the ${\rho_i}^{t_i}$s of that level. Hence, by line 16, agent A is set as the `tentative winner` and agent B and C and their subtrees are set as `non-winners`. Also, the `effective payment` of agent A to S is $\rho_B^{t_B/t_A} = 9^{3/1} = 729$. From line 19, the `actual`

Algorithm 1: `LbLEV`

Input: reported types $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i), \hat{r}_i \subseteq r_i, \; \forall i \in N$
Output: winner of the auction (which can be $\emptyset$), payments of each agent

1 **Preprocessing:** Since the underlying graph is a tree, let $\hat{T}$ be the sub-tree rooted at s induced from $\hat{r}_i, i \in N$. Pick $t \in \mathbb{R}^n_{>0}$ independent of the input
2 **if** $\hat{v}_i = 0, \forall\, i \in N$ **then**
3 Item is not sold and payment is set to zero for all agents, STOP
4 *Initialization:* all agents are `non-winners` and their `actual payments` are zeros, set `offset` = 0, `level` = 1, `parent` = s, $v_{\texttt{parent}} = 0$
5 In this level of $\hat{T}$:
6 **for** *each node* $i \in$ `children`(`parent`) **do**
7 Set `effective valuation` $\rho_i := \max\{\hat{v}_j : j \in \hat{T}_i\} -$ `offset`
8 Remove the nodes that have $\rho_i < 0$, denote the rest of the agents with $N_{\texttt{remain}}$
9 **if** $|N_{\texttt{remain}}| \geqslant 2$ **then**
10 Sort the nodes in decreasing order of $\rho_i^{t_i}$
11 Compute $z := \rho_\ell^{t_\ell/t_{i^*}}$, where i^* is the highest in this order and ℓ is the second highest node in the decreasing $\rho_i^{t_i}$ order
12 **else**
13 Set $z = 0$
14 **if** $v_{\texttt{parent}} \geqslant$ `offset` $+\, z$ **then**
15 STOP and go to Step 23
16 Set the highest node i^* in this order as the `tentative winner` and its `effective payment` to be z
17 All nodes and their subtrees except i^* are declared `non-winners`
18 The `actual payment` of i^* to `parent` = `effective payment` + `offset`
19 `parent` = i^*, `offset` = `actual payment` of i^*
20 `level` = `level` + 1
21 Repeat Steps 5 to 20 with the updated `parent` and `offset` for the new level
22 STOP when no agent i has $\rho_i \geqslant 0$ OR the leaf nodes are reached
23 Set `tentative winner` as `final winner`; final payments are the `actual payments` that are paid to the respective parents of $\hat{T}$

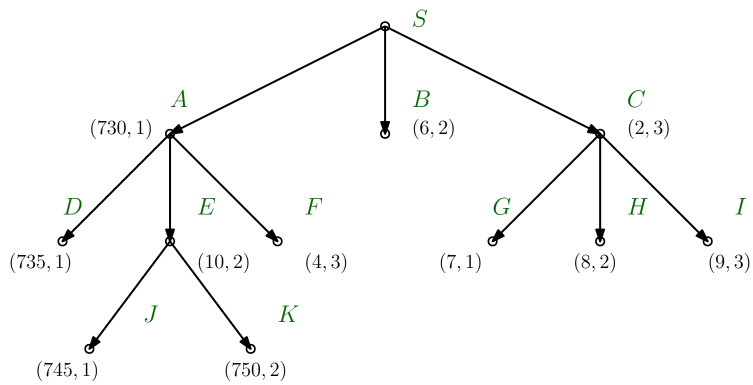

Figure 3: An example instance of `LbLEV`.

`payment` of A becomes the same as its `effective payment` since at this level `offset`$= 0$. Also, for the next iteration, i.e., for `level` = 2, `parent` $= A$ and `offset` $= 729$ are set.

At `level` = 2, $\rho_D = 735 - 729 = 6$, $\rho_E = 750 - 729 = 21$ and $\rho_F = 4 - 729 = -725$. As $\rho_F < 0$, F and its subtree is removed (line 8). Line 14 stands false, since $v_A = 730 \ngeqslant 729 + 6^{1/2} =$ `offset` $+ \rho_\ell^{t_\ell/t_{i^*}}$. Hence, E is the `tentative winner`, and it pays $729 + 6^{1/2} = 731.45$. Again from line 19, the next level (`level` = 3) details are updated, i.e., `parent` $= E$ and `offset` $= 731.45$.

At $\texttt{level} = 3$, $\rho_J = 745 - 731.45 = 13.55$, $\rho_K = 750 - 731.45 = 18.55$, Agent K becomes the `tentative winner` as line 14 returns false. The `actual payment` of K is $731.45 + 13.55^{1/2} = 735.13$.

As agent K is a leaf node, the algorithm stops via line 22 and K is declared as the `final winner` (line 23). Agent A gets the difference between the amounts it receives from its children and pays to its parent, i.e., $731.45 - 729 = 2.45$ (one can think of this amount as the *commission* to forward the information, which offsets its payoff when it manipulates and does not forward). Similarly, agent E gets 3.681 as the commission. The revenue generated by the auction is 729 which is the payment of agent A to S.

Note that, `LbLEV` actually defines a class of mechanisms. A specific instance of an `LbLEV` mechanism is identified by the chosen vector t. However, keeping with the tradition of the mechanism design literature, we call all such mechanisms `LbLEV`. For example, two instances of Groves mechanism may have different $h_i(\theta_{-i}), i \in N$, functions, but both of them are known by the same name. In the following results, we provide two important properties of `LbLEV`.

Theorem 4 `LbLEV` *is* `DDSIC`.

Remark: As discussed earlier in Section 4, in the classical single object auction characterization result by Myerson [22], the `VIPC` term could have been chosen independently. For every such choice, the mechanism would have been DSIC in that setting. In the case of an auction on the network, we need to additionally ensure the diffusion constraint (Eqn. (2)) that restricts the choice of the `VIPC`s. We show that the `VIPC`s corresponding to the `LbLEV` mechanism ensure it and this is where these mechanisms on the network are distinctly different from that of Myerson's characterization.

Before embarking on the proof, we want to give an overview since it is long and detailed and also define a few terminologies. The proof proceeds by showing that Alg. 1 satisfies all the three conditions of `MFF`, which is equivalent to `DDSIC` (Corollary 1). Given an instance of the reported types, we partition the agents into *three* classes that are exhaustive: (i) *winner* – agent who gets the item (`LbLEV` is deterministic, hence there will be a deterministic winner), (ii) *on-path non-winner* – agents that lie on the path from the seller to the winner, and (iii) *not-on-path non-winner* – agents that are not on the winning path from the seller to the winner. We show that the allocation function satisfies cond. (a) of `MFF` (Def. 5) for all these agents. The payments given by Alg. 1 matches Eqn. (1) for each of them with appropriate choices of the `VIPC` terms identified from the algorithm. The most crucial part of the proof is to show that the Eqn. (2) is also satisfied by the chosen `VIPC` terms. The `offset` chosen by the algorithm at each `level` is crucial. Updating the *effective* valuations of the nodes in the subtree of the current `parent` and elimination of the branch of the subtree that has all valuations smaller than the `offset`, ensure this critical diffusion constraint (Eqn. (2)).

Proof: [of Theorem 4] The proof is divided into three parts, each of which shows that `LbLEV` satisfies the three conditions of `MFF` (Def. 5). Since `LbLEV` is defined in a manner where sub-auctions happen in every level of the sub-tree $\hat{T}$ obtained from $\hat{G}$, we need a few terms defined for a cleaner presentation of the proof. These are:

- $\texttt{offset}(i)$: `offset` for the `children` of i, i.e., at a level where i is the auctioneer.
- $\texttt{children}(i)$ and $\texttt{parent}(i)$ are the `children` and `parent` of i respectively in $\hat{T}$, and is defined in the usual way for standard trees. Therefore, $\texttt{offset}(\texttt{parent}(i))$ will denote the `offset` set at one level before agent i by the `parent` of i, and will refer to the previous iteration of the `LbLEV` mechanism. Similarly, $\texttt{children}(\texttt{parent}(i))$ denotes the siblings of i including herself.
- $\texttt{winner}(i) := \text{argmax}_{j \in \texttt{children}(i)} \rho_j^{t_j}$, and $\texttt{runnerup}(i) := \text{argmax}_{j \in \texttt{children}(i) \setminus \texttt{winner}(i)} \rho_j^{t_j}$ denote the `winner` and `runnerup` respectively of the auction at a level where i is the auctioneer. Ties are broken arbitrarily in both these cases.

In the extreme case where the reported valuation of every agent is zero, the allocation probability is zero for every agent and so are their payments (according to line 2 of Alg. 1). Here it trivially satisfies all the three conditions of Def. 5. So, in the rest of the proof, we will assume that at least one agent has a positive reported valuation. Therefore, if some agent reports her valuation to be zero, she is not allocated the object.

Part 1: `LbLEV` **satisfies Condition (a) of** `MFF`**:** To show that the allocation function under `LbLEV` satisfies monotonicity w.r.t. the valuation of every agent, we need to show that for every pair v_i, v_i' s.t. $v_i' > v_i$, the allocation probability at v_i' is at least as much as at v_i, for all r_i (when agent i diffuses to r_i), and for all $i \in N$. Since the `LbLEV` mechanism is defined w.r.t. the effective valuations ρ_i's, and not v_i's, showing this is non-trivial. Based on the fact that a typical agent i can belong to one of the three classes after the outcome of the mechanism is chosen, we have the following cases when agent i reports a valuation of v_i.

Case 1: agent i is a *not-on-path non-winner*: in this case, $g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 0$. From the description of `LbLEV`, it is clear that for $v_i' > v_i$, either agent i can remain a *not-on-path non-winner*, or it can become a *winner*. It cannot become an *on-path non-winner* because it would imply that there was another agent in i's subtree that had a maximum valuation in this network at agent i's original valuation v_i and then agent i could not be a *not-on-path non-winner*. In both the cases where agent i is *not-on-path non-winner* or *winner*, $g_i(f^G(v_i', r_i), \hat{\theta}_{-i}) \geqslant 0$, hence it is monotone non-decreasing.

Case 2: agent i is an *on-path non-winner*: the allocation for agent i is $g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 0$ in this case as well. This agent is *on-path non-winner* with bid v_i implies that there is an agent in $\hat{T}_i$ that has reported the winning bid. Now, if agent i bids v_i' which is higher than v_i, it can either continue to be an *on-path non-winner* or may become the new *winner* at a sufficiently high bid. In both these cases, the allocation probability is monotone non-decreasing.

Case 3: agent i is the *winner*: here $g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 1$. We need only to show that for all $v_i' > v_i$, agent i continues to be the winner. This is fairly easy to see from Alg. 1. An agent can be the winner either when it is the parent node in line 14 or line 22.

In line 14, since agent i is the parent and it satisfies the `if` condition of that line, an increase in its valuation will continue to hold that condition true and i will continue to be the *winner*.

In line 22, agent i is the auctioneer whose `offset` is higher than the valuations of all agents in its subtree. The `offset` is not a function of agent i's valuation – it is determined by $\rho_\ell^{t_\ell/t_i}$, where ℓ is the second highest node in the decreasing $\rho_k^{t_k}$ order (line 13). Hence, an increase in v_i will continue keeping i to be the winner.

These three cases together prove this part of the proof.

Part 2: `LbLEV` **satisfies Condition (b) of** `MFF`**:** To show this part of the proof, we need to show that the payments given by Alg. 1 for each agent matches Eqn. (1). Note that, after a monotone non-decreasing allocation rule has been picked (as seen in the previous case of this proof) by the algorithm, the only variable quantity in the payment formula is the `VIPC` term. The other two terms, i.e., the second and third term on the RHS of Eqn. (1) are already fixed given the allocation. So, in order to complete the proof, we need to find an appropriate `VIPC` such that the payment given by `LbLEV` exactly matches the sum of those two terms and the `VIPC`.[3] We will denote the payment for agent i given by the mechanism as $p_i^{\texttt{LbLEV}}((v_i, r_i'), \hat{\theta}_{-i})$ when the reported type of agent i is (v_i, r_i') and that for the other agents are $\hat{\theta}_{-i}$.

Case 1: agent i is a *not-on-path non-winner*: note that, in this case, $g_i(f^G(v_i, r_i'), \hat{\theta}_{-i}) = 0$. Hence, the last two terms in the RHS of Eqn. (1) vanishes. We need to set that the $\texttt{VIPC}_i$ term which is exactly equal to $p_i^{\texttt{LbLEV}}((v_i, r_i'), \hat{\theta}_{-i})$, the actual payment of agent i under `LbLEV`, and show that it is indeed independent of v_i for it to be qualified as a `VIPC`. From the algorithm, we see that $p_i^{\texttt{LbLEV}}((v_i, r_i'), \hat{\theta}_{-i}) = 0$, since i is a *not-on-path non-winner* agent. Hence, $\texttt{VIPC}_i = 0$, and it matches conditions (c) and (b) of Def. 5.

Case 2: agent i is an *on-path non-winner*: for this case as well, the situation is similar to the previous case: $g_i(f^G(v_i, r_i'), \hat{\theta}_{-i}) = 0$, and hence, the last two terms in the RHS of Eqn. (1) vanishes. We need to calculate $p_i^{\texttt{LbLEV}}((v_i, r_i'), \hat{\theta}_{-i})$, the actual payment of agent i under `LbLEV`, and show that it is indeed independent of v_i for it to be qualified as $\texttt{VIPC}_i$.

[3] Since we discuss the values of the `VIPC`s at a specific instance, i.e., after the variables $r_i', \hat{\theta}_{-i}$ have realized, the arguments of those `VIPC` terms are clear from the context. Therefore, we will omit the arguments of the `VIPC` terms for brevity in the rest of the paper.

From the LbLEV algorithm, we see that the net payment of a *on-path non-winner* agent i has the following simple structure:

$$p_i^{\texttt{LbLEV}}((v_i, r'_i), \hat{\theta}_{-i}) = \pi(\hat{T}_i) - \sum_{j \in \texttt{children}(i)} \pi(\hat{T}_j) =: \pi(\hat{T}_i) - R_i, \; \forall r'_i \subseteq r_i \tag{10}$$

Where $\pi(\hat{T}_k)$ is payment made by the subtree rooted at k to its parent (called the `actual payment` in Alg. 1). Therefore, $\sum_{j \in \texttt{children}(i)} \pi(\hat{T}_j)$ is the net payment received by agent i from the subtrees rooted at its children nodes. Therefore, the payment of agent i is just the difference between the payment it makes to $\texttt{parent}(i)$ minus the sum of the payments it receives from $\texttt{children}(i)$ (according to line 23). We use the shorthand R_i to denote $\sum_{j \in \texttt{children}(i)} \pi(\hat{T}_j)$.

Now, we need to show that the RHS of Eqn. (10) is independent of v_i and then we are done claiming it to be $\texttt{VIPC}_i$. From the algorithm, we find that the payment received by i has a rather simpler form:

$$R_i = \texttt{offset}(i) + \rho_\ell^{t_\ell / t_k}, \text{ where } k = \texttt{winner}(i), \ell = \texttt{runnerup}(i). \tag{11}$$

The above equation means that the payment received by i is the `actual payment` the winner of this level, k, makes to its parent i, and it is the sum of two terms: (a) the offset set by i while auctioning at that level and (b) the ρ of the second highest bidder in the decreasing order of $\rho_i^{t_i}$ raised to an appropriate exponent.

From line 19, we find that $\texttt{offset}(i) = \pi(\hat{T}_i)$, i.e., the payment the winning agent i of a level makes to its parent is set as the `offset` in the next level. Note that, there must be at least one next level since it is an *on-path non-winner*. Therefore, the RHS of Eqn. (10) becomes $-\rho_\ell^{t_\ell / t_k}$. We claim that this is independent of v_i. The exponents are constants and independent of v_i. The term $\rho_\ell = v_\ell - \texttt{offset}(i)$ is also independent of v_i. This is because the first term v_ℓ is independent of v_i. The second term $\texttt{offset}(i)$ is the payment $\pi(\hat{T}_i)$, that i made to its parent which is a function of the $\texttt{offset}(\texttt{parent}(i))$ and the valuations of the agents other than agent i in the level where $\texttt{parent}(i)$ was the auctioneer and i was a participant of that auction.

$$\pi(\hat{T}_i) = \texttt{offset}(\texttt{parent}(i)) + \rho_j^{t_j / t_i}, \text{ where } j = \texttt{runnerup}(\texttt{parent}(i)).$$

Since, both these terms are independent of v_i, the RHS of Eqn. (10) is independent of v_i and hence it is a valid $\texttt{VIPC}_i$.

Case 3: agent i is the *winner*: unlike the previous two cases, here $g_i(f^G(v_i, r'_i), \hat{\theta}_{-i}) = 1$. Therefore, the payment given by Eqn. (1) has the last two terms on the RHS that are non-zero. The first term of the RHS, i.e., the VIPC, is the payment of the agent when it reports 0 instead of v_i. From the algorithm, we observe that an agent i can become a winner in two possible ways: (i) if i's valuation is larger than the maximum payment it can extract from $\texttt{children}(i)$ in $\hat{T}_i$ (line 14), or (ii) if i's offset is so high that none of its children has a positive `effective valuation` ρ or i is a leaf node (line 22). In the first case, if agent i reports a valuation of 0, then it becomes a *on-path non-winner*, and in the second, it becomes a *not-on-path non-winner*. In the following, we consider these two cases separately and show that $p_i^{\texttt{LbLEV}}((v_i, r'_i), \hat{\theta}_{-i})$ indeed matches the expression of Eqn. (1).

- *Case 3(i):* bidding 0 makes agent i a *on-path non-winner*: when agent i is an *on-path non-winner*, its payment is $\texttt{VIPC}_i = -\rho_\ell^{t_\ell / t_k}$, where $k = \texttt{winner}(i), \ell = \texttt{runnerup}(i)$ (from Case 2 above). Given the allocation function is already fixed via the LbLEV mechanism, the last two terms in Eqn. (1) can be written w.r.t. that allocation as

$$\begin{aligned}
& g_i(f^G((v_i, r'_i), \hat{\theta}_{-i}))v_i - \int_0^{v_i} g_i(f^G((v, r'_i), \hat{\theta}_{-i}))dv \\
&= g_i(f^G((v_i, r'_i), \hat{\theta}_{-i}))v_i - \int_0^{l_i} g_i(f^G((v, r'_i), \hat{\theta}_{-i}))dv - \int_{l_i}^{v_i} g_i(f^G((v, r'_i), \hat{\theta}_{-i}))dv \\
&= 1 \cdot v_i - \int_0^{l_i} 0 \cdot dv - \int_{l_i}^{v_i} 1 \cdot dv = l_i,
\end{aligned}$$

  where l_i is the threshold after which agent i starts becoming the winner. The *winner* of the LbLEV mechanism is deterministic and this agent starts becoming the *winner* when its valuation crosses a threshold point that we define to be l_i, which is guaranteed to exist. Now, we see that agent i, which was an *on-path non-winner*, starts becoming a *winner* only when it is the `parent` in line 14 of the algorithm. Hence the threshold l_i will be as follows.

$$l_i = \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k}, \text{ where } k = \texttt{winner}(i), \ell = \texttt{runnerup}(i).$$

Therefore, the entire payment given by Eqn. (1) is $\texttt{VIPC}_i + l_i = \texttt{offset}(i)$. From the algorithm, we notice that the *winner* pays its own `offset` to its parent. Therefore,

$$p_i^{\texttt{LbLEV}}((v_i, r_i'), \hat{\theta}_{-i}) = \texttt{offset}(i)$$

Hence, we have the equality of the `LbLEV` payment with that of Eqn. (1).

- *Case 3(ii):* bidding 0 makes agent i a *not-on-path non-winner*: when agent i is a *not-on-path non-winner*, its payment is $\texttt{VIPC}_i = 0$ (from Case 1 above). Given the allocation function is already fixed via the `LbLEV` mechanism, the last two terms in Eqn. (1) can be written w.r.t. that allocation as

$$\begin{aligned} & g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))v_i - \int_0^{v_i} g_i(f^G((v, r_i'), \hat{\theta}_{-i}))dv \\ &= g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))v_i - \int_0^{k_i} g_i(f^G((v, r_i'), \hat{\theta}_{-i}))dv - \int_{k_i}^{v_i} g_i(f^G((v, r_i'), \hat{\theta}_{-i}))dv \\ &= 1 \cdot v_i - \int_0^{k_i} 0 \cdot dv - \int_{k_i}^{v_i} 1 \cdot dv = k_i, \end{aligned}$$

where k_i is the threshold after which agent i starts becoming the winner. The *winner* of the `LbLEV` mechanism is deterministic and this agent starts becoming the *winner* when its value crosses a threshold point that we define to be k_i, which is guaranteed to exist. Since, in this case, i was a *not-on-path non-winner* till its value reached k_i, this critical valuation is given by

$$k_i = \texttt{offset}(\texttt{parent}(i)) + \rho_j^{t_j/t_i}, \text{ where } j = \texttt{runnerup}(\texttt{parent}(i)).$$

If v_i crosses the value on the RHS above, then ρ_i becomes the maximum among the children of $\texttt{parent}(i)$, and hence it becomes the winner.
On the other hand, under this case, i becomes the winner because line 22 of the `LbLEV` mechanism becomes effective. Hence, we have

$$p_i^{\texttt{LbLEV}}((v_i, r_i'), \hat{\theta}_{-i}) = \texttt{offset}(\texttt{parent}(i)) + \rho_j^{t_j/t_i}, \text{ where } j = \texttt{runnerup}(\texttt{parent}(i))$$

Since $\texttt{VIPC}_i = 0$ in this case, we have the equality of the `LbLEV` payment with that of Eqn. (1).

Part 3: `LbLEV` satisfies Condition (c) of `MFF`: In this condition, we need to compare $\texttt{VIPC}_i$ between the cases when i forwards to its complete neighbor set (r_i) versus a subset of its neighbors ($r_i' \subseteq r_i$). By reporting a diffusion type, agent i can be in one of the three classes: *not-on-path non-winner*, *on-path non-winner*, or *winner*. We handle these cases one by one.

Case 1: agent i is a not-on-path non-winner *when it diffuses to complete neighbor set r_i:*, then either all nodes from i to the root s on $\hat{T}$ were never tentative winners, or some node in that path is the winner. In both cases, if agent i does not forward the information to r_i and instead diffuses to r_i', s.t. $r_i' \subseteq r_i$, the winner does not change. Hence, $\texttt{VIPC}_i = 0$ and $g_i = 0$ for any diffusion type. Therefore, Eqn. (2) is trivially satisfied.

Case 2: agent i is the winner *node when it diffuses to complete neighbor set r_i:* then `LbLEV` already treats i as if it does not forward and calculates $\texttt{VIPC}_i$. So, $\texttt{VIPC}_i$ of agent i at diffusion type r_i is the same as when i forwards to r_i', some subset of r_i. Also, i will continue to be the *winner* for any diffusion type. Hence $g_i = 1$ for any diffusion type reported by agent i. Therefore, similar to *Case 1*, Eqn. (2) is trivially satisfied.

Case 3: agent i is an on-path non-winner *node when it diffuses to entire neighbor set r_i:* this is a non-trivial case, since by strategic forwarding, agent i may change the winner. By partial or not forwarding, an *on-path non-winner* agent i can either become a *not-on-path non-winner* or a *winner*.

- When i becomes a *not-on-path non-winner* by diffusing to $r_i' \subset r_i$, its utility becomes 0 (see *Case 1* of Part 2 above). However, an *on-path non-winner* draws utility $R_i - \pi(\hat{T}_i)$ where $R_i = \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k} = \pi(\hat{T}_i) + \rho_\ell^{t_\ell/t_k}$ (see *Case 2* of Part 2 above). As $\rho_\ell \geqslant 0$, hence utility being an *on-path non-winner* is non-negative and makes diffusion to r_i a weakly better option for i.
- In the other case, when i becomes a *winner* by strategic forwarding, its utility becomes $v_i - \pi(\hat{T}_i)$. Note that i is an *on-path non-winner* because it failed the `if` condition in line 14, hence,

$$v_i < \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k}, \text{ where } k = \texttt{winner}(i), \ell = \texttt{runnerup}(i). \tag{12}$$

  Now, as an *on-path non-winner*, i's utility is $R_i - \pi(\hat{T}_i)$ (since the allocation probability of i is zero by Eqn. (10)). However, the actual payment by $\texttt{children}(i)$ to i is

$$R_i = \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k}, \text{ where } k = \texttt{winner}(i), \ell = \texttt{runnerup}(i). \tag{13}$$

  Therefore, from Eqns. (12) and (13) we get, $v_i < R_i$. Agent i pays $\pi(\hat{T}_i)$ to its parent regardless of whether it forwards to r_i or not. Therefore, if agent i forwards to r_i, it gets an utility of $R_i - \pi(\hat{T}_i) = \rho_\ell^{t_\ell/t_k} \geqslant 0$ which is larger than the utility $v_i - \pi(\hat{T}_i)$ of i when it diffuses to some $r_i' \subset r_i$ and becomes the *winner*. Therefore, in this case as well, forwarding to r_i is better than any partial forwarding to $r_i' \subset r_i$ for agent i.

Since agent i and $r_i' \subseteq r_i$ is arbitrary, we conclude that forwarding to the entire neighbor set r_i is a weakly dominant strategy than forwarding to a subset r_i' for every i in each of the three cases above. This gives,

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$$

Since, we have already shown in the previous part of this proof that the payment expression is given by Eqn. (1), expanding $p_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ and $p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$ according to that expression in the above inequality, we get

$$p_i(f^G((0, r_i'), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) \geqslant \int_0^{v_i} \Big(g_i(f^G((y, r_i'), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \Big)\, dy.$$

Hence, Eqn. (2) holds for `LbLEV`.

Hence, combining the Parts 1 to 3, we conclude that `LbLEV` is `DDSIC`. ∎

Theorem 5 `LbLEV` *is IR.*

Proof: We need to show that for every agent $i \in N$, the net utility $v_i - p_i^{\texttt{LbLEV}}((v_i, r_i), \hat{\theta}_{-i}) \geqslant 0$. We know that for a reported type profile $((v_i, r_i), \hat{\theta}_{-i})$, agent i can be one of the following.

- Agent i is *not-on-path non-winner*: in this case, i has an allocation probability of zero, since it never becomes a `tentative winner`. Also, by Alg. 1, its payment remains zero throughout. Therefore, such an agent satisfies the IR condition (Def. 4) trivially.
- Agent i is *on-path non-winner*: in this case, i's allocation probability is still zero, and it makes a payment of $\pi(\hat{T}_i) - R_i$ as given by Eqn. (10). In the discussion following that equation, we see that $R_i = \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k}$, where $k = \texttt{winner}(i), \ell = \texttt{runnerup}(i)$, (Eqn. (11)) and that $\texttt{offset}(i) = \pi(\hat{T}_i)$. Therefore, the utility of agent i in this case is $0 - \pi(\hat{T}_i) + R_i = \rho_\ell^{t_\ell/t_k} \geqslant 0$.
- Agent i is *winner*: from the algorithm, we observe that an agent i can become a winner in two possible ways: (i) if i's valuation is larger than the maximum payment it can extract from $\texttt{children}(i)$ in $\hat{T}_i$ (line 14), or (ii) if i's offset is so high that none of its children has a positive `effective valuation` ρ or i is a leaf node (line 22). However, in both the cases, the payment of the agent is given by $\texttt{offset}(\texttt{parent}(i)) + \rho_j^{t_j/t_i}$, where $j = \texttt{runnerup}(\texttt{parent}(i))$.

  Also, agent i is the winner implies that its ρ_i is such that $\rho_i^{t_i}$ is the largest among all its siblings, i.e., $\texttt{children}(\texttt{parent}(i))$. Therefore, we can write

$$\rho_i^{t_i} \geqslant \rho_j^{t_j}, \text{ where } j = \texttt{runnerup}(\texttt{parent}(i))$$

$$\Rightarrow \quad \rho_i \geqslant \rho_j^{t_j/t_i}, \text{ since } \rho_i, t_i > 0$$
$$\Rightarrow \quad v_i \geqslant \texttt{offset}(\texttt{parent}(i)) + \rho_j^{t_j/t_i} = p_i^{\texttt{LbLEV}}((v_i, r_i), \hat{\theta}_{-i})$$
$$\Rightarrow \quad v_i - p_i^{\texttt{LbLEV}}((v_i, r_i), \hat{\theta}_{-i}) \geqslant 0.$$

The second implication follows from the fact that LbLEV sets ρ_i to be the difference between the reported valuation of i and the offset set by its parent node. Therefore, the RHS becomes the payment of agent i.

Considering the three cases above, we conclude that LbLEV is IR. ■

Why LbLEV? The illustration of LbLEV serves two purposes: (a) it shows that an IC network auction can have quite high diversity in its design, and (b) this diversity can be exploited in order to earn higher revenue from an auction. In the experiments (Section 7), we show this via simulations. However, tuning the mechanism for a higher revenue requires prior information on the valuations.

6 Bayesian Setup and Optimal Auction

The optimal auction is the one that maximizes the expected revenue. This is done assuming that the prior of the valuations are known to the auctioneer, which is a common assumption in classical auction literature [22, e.g.].[4] In this section, we consider the revenue-optimal auction where the prior distribution over (v_i, v_{-i}) is given by P and is a common knowledge. First, we define the notion of truthfulness in the prior-based setup.

Definition 6 (Diffusion Bayesian Incentive Compatibility) *A* DA (g, p) *on a graph* G *is* diffusion Bayesian incentive compatible (DBIC) *if*

1. *every agent's expected utility is maximized by reporting her true valuation irrespective of the diffusing status of herself and the other agents, i.e., for every* $i \in N$, $\forall r_i, \hat{r}_{-i}$, *the following holds*

$$\mathbb{E}_{v_{-i}|v_i}\left[v_i g_i(f^G((v_i, r'_i), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i, r'_i), (v_{-i}, \hat{r}_{-i})))\right]$$
$$\geqslant \mathbb{E}_{v_{-i}|v_i}\left[v_i g_i(f^G((v'_i, r'_i), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v'_i, r'_i), (v_{-i}, \hat{r}_{-i})))\right], \forall v_i, v'_i, r'_i \subseteq r_i, \text{ and,}$$

2. *for every true valuation, every agent's expected utility is maximized by diffusing to all its neighbors irrespective of the diffusion status of the other agents, i.e., for every* $i \in N$, $\forall r_i, \hat{r}_{-i}$, *the following holds*

$$\mathbb{E}_{v_{-i}|v_i}\left[v_i g_i(f^G((v_i, r_i), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i, r_i), (v_{-i}, \hat{r}_{-i})))\right]$$
$$\geqslant \mathbb{E}_{v_{-i}|v_i}\left[v_i g_i(f^G((v_i, r'_i), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i, r'_i), (v_{-i}, \hat{r}_{-i})))\right], \forall v_i, r'_i \subseteq r_i.$$

It is easy to see that DDSIC implies DBIC since DBIC requires Conditions 1 and 2 of Def. 2 to hold only in expectation.

6.1 Characterization of DBIC Mechanisms

Our first result is to characterize the DBIC auctions. For convenience, we define the *expected* allocation and payments with the shorthand notation as described below.

$$\alpha_i((v_i, r'_i), \hat{r}_{-i})) = \mathbb{E}_{v_{-i}|v_i}[g_i(f^G((v_i, r'_i), (v_{-i}, \hat{r}_{-i})))] \qquad \textbf{(allocation)} \tag{14}$$
$$\texttt{pay}_i((v_i, r'_i), \hat{r}_{-i})) = \mathbb{E}_{v_{-i}|v_i}[p_i(f^G((v_i, r'_i), (v_{-i}, \hat{r}_{-i})))] \qquad \textbf{(payment)} \tag{15}$$

In the Bayesian setup, the notion of participation guarantee is also weakened to *interim individual rationality (IIR)* where the expected utility of a player to join the mechanism is non-negative after she learns her own type.

Definition 7 (Interim Individual Rationality) *A* DA (g, p) *on a graph* G *is* interim individually rational (IIR) *if* $v_i \alpha_i((v_i, r_i), \hat{r}_{-i})) - \texttt{pay}_i((v_i, r_i), \hat{r}_{-i})) \geqslant 0$, $\forall v_i, \hat{\theta}_{-i}, r_i, \forall i \in N$, *where* r_i *is the true neighbor set of* i.

[4] This assumption is primarily due to two reasons: (a) for prior-free auctions, the worst-case revenue can be arbitrarily bad, hence revenue maximization does not yield any useful result, and (b) in practice, the prior on the users' valuation can be estimated from the historical data.

Similar to DBIC, it is easy to see that IR implies IIR since IIR requires the conditions of Def. 4 to hold only in expectation. Next, we define the following structure of an auction to succinctly characterize the DBIC auctions.

Definition 8 (Monotone and Forwarding-Friendliness in Expectation (MFFE)) *For a given network G, a* DA (g, p) *is* monotone and forwarding-friendly in expectation (MFFE) *if*

(a) for every $i \in N$ and $r_i, \hat{r}_{-i}$, the functions $\alpha_i((v_i, r_i'), \hat{r}_{-i})$ is non-decreasing in v_i, for every $r_i' \subseteq r_i$, and for the given allocation function α, the payment $\mathtt{pay}_i$ for each player $i \in N$ is such that, for every v_i, r_i, and $\hat{r}_{-i}$, the following two conditions hold.

(b) For every $r_i' \subseteq r_i$, the following payment formula is satisfied.

$$\mathtt{pay}_i((v_i, r_i'), \hat{r}_{-i}) = \mathtt{pay}_i((0, r_i'), \hat{r}_{-i}) + v_i \alpha_i((v_i, r_i'), \hat{r}_{-i}) - \int_0^{v_i} \alpha_i((y, r_i'), \hat{r}_{-i}) \, \mathrm{d}y \quad (16)$$

(c) The values of $\mathtt{pay}_i((0, r_i'), \hat{r}_{-i})$ and $\mathtt{pay}_i((0, r_i), \hat{r}_{-i})$ are arbitrary real numbers that satisfies the following inequality for every $r_i' \subseteq r_i$.

$$\mathtt{pay}_i((0, r_i'), \hat{r}_{-i}) - \mathtt{pay}_i((0, r_i), \hat{r}_{-i}) \geqslant \int_0^{v_i} (\alpha_i((y, r_i'), \hat{r}_{-i}) - \alpha_i((y, r_i), \hat{r}_{-i})) \, \mathrm{d}y \quad (17)$$

Theorem 6 (DBIC Characterization) *A* DA (g, p) *is* DBIC *if and only if it is* MFFE.

Proof sketch: The proof of the direction MFFE $\Rightarrow$ DBIC is identical to Theorem 2 with the allocations and payments, g_i and p_i, replaced with their expected versions, α_i and $\mathtt{pay}_i$ (Eqns. (14) and (15)), respectively. The other direction is almost identical to Theorem 1 since the same starting inequalities also hold for DBIC with the above-mentioned replacements, i.e., g_i and p_i, replaced with their expected versions, α_i and $\mathtt{pay}_i$ (Eqns. (14) and (15)), respectively. We skip rewriting the almost identical steps with the above-mentioned substitutions. ■

6.2 Referral auctions

Multi-level marketing (MLM) is a marketing approach that incentivizes individuals who not only adopt a product but advertise it also [1, 7, 8]. On a social network, it creates a viral effect where the information regarding a product reaches far beyond what traditional marketing can do. Due to its similarity with the objective of diffusion auctions, i.e., to spread the information of the auction to more individuals on a network, in this section, we consider a natural adaptation of MLM into auctions and call this class of auctions as *referral auctions* (RA).

In a referral auction, the seller invites its immediate neighbors in the network to report their valuations and invite all of their neighbors. These agents are also suggested, in turn, to spread the same message, i.e., to ask their neighbors to report their valuations and forward the information to their neighbors. Each time a node i reports and forwards the information to its neighbors, the information of $(\hat{v}_i, \hat{r}_i)$ is recorded by the seller along with its (system-generated) timestamp τ_i. Note that one can implement this auction in various possible ways, e.g., via inviting each node to register on the seller's site and providing the information of their neighbors. In all possible such cases, the seller can record the timestamp which cannot be manipulated by the agents. This information will be used by the class of referral auctions.

In the class RA, all agents are sorted w.r.t. their timestamps and a referral tree is formed via a first-invite-first-served policy (breaking ties in a fixed order, e.g., w.r.t. their social IDs). This implies that the unique parent of every node is determined by the earliest timestamp of those inviting nodes. Note that, this is the principle of multi-level marketing as well – only those individuals on a network are considered for referral bonuses that invited a new customer *first* to the seller's system.

Once the referral tree is formed, the mechanism runs an auction at every level of the tree through a general *deterministic* allocation rule $\mathscr{g}$ which is monotone non-decreasing and runs only on the agents at a given level. Define the corresponding payment as

$$\mathscr{p}_i(\rho_i, \rho_{-i}) = \rho_i \mathscr{g}_i(\rho_i, \rho_{-i}) - \int_0^{\rho_i} \mathscr{g}_i(y, \rho_{-i}) \, \mathrm{d}y. \quad (18)$$

Algorithm 2: Referral Auctions (RA)

Input: reported types $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i), \hat{r}_i \subseteq r_i$, and recorded timestamps τ_i, for all $i \in N$
Parameter : an arbitrary monotone non-decreasing deterministic allocation $\mathscr{g}$
Output: winner of the auction (which can be $\emptyset$), payments of each agent

1 **Preprocessing:** Create the referral tree $\hat{T}$ rooted at s such that the neighbors of s is $\texttt{children}(s)$, and $\texttt{parent}(i) = \operatorname{argmin}\{\tau_k : i \in \hat{r}_k\}$, for all $i \in N \setminus \texttt{children}(s)$. Ties are broken w.r.t. a fixed order over the nodes.
2 **if** $\hat{v}_i = 0, \forall\, i \in N$ **then**
3   Item is not sold and payment is set to zero for all agents, STOP
4 *Initialization:* all agents are `non-winners` and their `actual payments` are zeros, set `offset` = 0, `level` = 1, `parent` = s, $v_{\texttt{parent}} = 0$
5 In this level of $\hat{T}$:
6   **for** *each node* $i \in \texttt{children}(\texttt{parent})$ **do**
7   Set `effective valuation` $\rho_i := \max\{\hat{v}_j : j \in \hat{T}_i\} - \texttt{offset}$
8   Remove the nodes that have $\rho_i < 0$, denote the rest of the agents with $N_{\texttt{remain}}$
9   **if** $|N_{\texttt{remain}}| \geqslant 2$ **then**
10   Find i^* where $\mathscr{g}_{i^*}(\rho_{i^*}, \rho_{N_{\texttt{remain}} \setminus \{i^*\}}) = 1$
11   Compute $z := \mathscr{p}_{i^*}(\rho_{i^*}, \rho_{N_{\texttt{remain}} \setminus \{i^*\}})$, given by Eqn. (18)
12   **else**
13   Set $z = 0$
14   **if** $v_{\texttt{parent}} \geqslant \texttt{offset} + z$ **then**
15   STOP and go to Step 23
16   Set agent i^* as the `tentative winner` and its `effective payment` to be z
17   All nodes and their subtrees except i^* are declared `non-winners`
18   The `actual payment` of i^* to `parent` = `effective payment` + `offset`
19   `parent` = i^*, `offset` = `actual payment` of i^*
20   `level` = `level` + 1
21 Repeat Steps 5 to 20 with the updated `parent` and `offset` for the new level
22 STOP when no agent i has $\rho_i \geqslant 0$ OR the leaf nodes are reached
23 Set `tentative winner` as `final winner`; final payments are the `actual payments` that are paid to the respective parents of $\hat{T}$

We note that the payment formula in Eqn. (18) is the same as the payment formula in the classical result of Myerson [22] with the VIPC term being zero. Based on different choices of $\mathscr{g}$, we obtain the class RA, described algorithmically in Alg. 2.

We show that each member of RA also follows the desirable properties like LbLEV. Since the proof is quite similar to that of LbLEV, we provide the sketch to show exactly the places where the proof differs. Note that the mechanisms in RA generate a referral tree $\hat{T}$ from an arbitrary underlying network. Hence, to prove truthfulness of the auctions in this class, we need to show that no agent can profit by underreporting her set of *true* neighbors in the underlying graph.

Theorem 7 *In each auction in* RA*, no agent* $i \in N$ *gets a higher utility by reporting* $\hat{r}_i \subset r_i$.

Proof: Each auction in RA is designed in such a way that only the *on-path non-winner* or the *winner* gets a non-negative utility. Each *not-on-path non-winner* node gets a utility of zero. Also, note that the auctions in RA create the referral tree in a first-invite-first-served manner. Since the agents cannot alter their timestamps, if they under-report their neighbor set, they can potentially stop becoming a *on-path non-winner*, which does not improve their utility. This observation is the key to this proof. Consider the following four cases for an agent i.

Case 1: agent i *is the* winner. In this case, her forwarding information is irrelevant to her utility. Hence, it does not violate the condition of the theorem.

Case 2: agent i *is a* not-on-path non-winner *after reporting* r_i*, her true neighbor set.* In this case as well, her forwarding information is irrelevant to her utility. This is because, even after reporting her entire neighbor set, she was not on the path to the winner. By misreporting, agent i will continue to be a *not-on-path non-winner* and her utility will continue to be zero.

Case 3: agent i is an on-path non-winner *when it reports r_i, but a* not-on-path non-winner *when it reports $r'_i \subset r_i$.* In this case, the utility is zero when agent i is a *not-on-path non-winner*, but her utility is non-negative when she is an *on-path non-winner*. Therefore, agent i cannot improve her utility in this case as well.

Case 4: agent i is an on-path non-winner *when it reports r_i, and continues to be a* on-path non-winner *when it reports $r'_i \subset r_i$.* According to Alg. 2, agent i gets the same utility in both these cases.

Hence, we conclude that in all possible cases of misreported neighbor set that alter the referral tree $\hat{T}$, an agent cannot obtain a better utility. ∎

Theorem 8 *Each auction in* RA *is* DDSIC *and IR.*

Proof sketch: Consider an arbitrary auction $f \in \texttt{RA}$. We showed in Theorem 7 that an agent cannot manipulate the referral tree to her favor. Hence, we need to show that for the formed referral tree $\hat{T}$, f satisfies DDSIC and IR. The proof follows similar line of arguments as Theorems 4 and 5 with a few variations, which we describe here. We follow the same definitions of *on-path non-winner*, *not-on-path non-winner*, and *winner* for the different types of agents, and use the terms `offset`, `children`, and `parent` as defined there. The `winner` function is updated as $\texttt{winner}(i) = \arg_{j \in \texttt{children}(i)} \{g_j(\rho_j, \rho_{N_{\texttt{remain}} \setminus \{j\}}) = 1\}$, and there is no `runnerup` function.

Part 1: f satisfies Condition (a) of MFF (Def. 5): This part follows by the same arguments and the fact that g is monotone non-decreasing.

Part 2 and 3: f satisfies Conditions (c) and (b) of MFF (Def. 5): These two arguments ensure the payment formula and the condition on the VIPC terms. The same conditions can be obtained with the same set of arguments by replacing $\rho_\ell^{t_\ell/t_k}$ with $\inf\{\rho_i \in \mathbb{R} : g_i(\rho_i, \rho_{N_{\texttt{remain}} \setminus \{i\}}) = 1\}$ at every level of the tree. Since ρ_i is obtained by subtracting $\texttt{offset}(\texttt{parent}(i))$ from agent i's valuation and that agent is removed if this number is negative, ρ_i's are non-negative by design. Hence, the number $\inf\{\rho_i \in \mathbb{R} : g_i(\rho_i, \rho_{N_{\texttt{remain}} \setminus \{i\}}) = 1\}$ is also non-negative. Therefore, this number follows every argument that $\rho_\ell^{t_\ell/t_k}$ followed at every level of the proof of Theorem 4 in an identical way.

Collecting these three parts, we prove that f is DDSIC.

Similarly, the IR proof follows an identical set of arguments follow with the same substitution of $\rho_\ell^{t_\ell/t_k}$ with $\inf\{\rho_i \in \mathbb{R} : g_i(\rho_i, \rho_{N_{\texttt{remain}} \setminus \{i\}}) = 1\}$ at every level of the tree. We skip rewriting the identical steps with the above-mentioned substitution. ∎

Since each $f \in \texttt{RA}$ is DDSIC and IR, they are DBIC and IIR. For simplicity of terminology, we will call each member of the class RA simply an RA (referral auction) henceforth.

In what follows, we will find an RA that maximizes the expected revenue of the seller when the valuations of the buyers are i.i.d. The high-level idea of our proof is the following. We observe from Eqn. (18) that the revenue of the seller in any auction in RA is simply the sum of the payments made by the buyers at the first level, when their valuations are replaced with their effective valuations, i.e., the maximum valuation in their subtree. This allows us to "replace" each buyer at the first level, with the buyer having a maximum valuation in its subtree. We formalize this idea in the proof of Theorem 9 and Lemma 1.

6.3 Optimal referral auction for i.i.d. valuations

In the objective of finding the *revenue-optimal* mechanism on a network, we address the problem in steps. In this section, we consider the mechanisms in the class RA, assuming that the priors on the valuations are known to the designer and that all v_i's are i.i.d. with distribution F that follows the *monotone hazard rate* (MHR) condition, i.e., $f(x)/(1 - F(x))$ is non-decreasing in x.[5] In

[5]The intuitive meaning of this condition is that the distribution is not *heavy-tailed*. Many distributions, e.g., uniform and exponential, follow the MHR condition [2].

this section, we find the optimal mechanism for this setup. To do that, first, we need to define a *transformed auction* (TA) of an RA as follows.

Definition 9 (Transformed Auction) *A* transformed auction *(*TA*) of an* RA *is the auction where each subtree* $\hat{T}_i, i \in \texttt{children}(s)$ *is replaced with a node with a valuation of* $\max_{j \in \hat{T}_i} v_j$*, and the allocation and payments are given by* $(\mathscr{g}_i, \mathscr{p}_i), i \in \texttt{children}(s)$.

Note that a TA does not specify the allocations beyond the first level of the tree. This is because, we will only be interested in the revenue generated by a TA, and every TA, regardless of how it allocates the object and extracts payments in the subsequent levels, will earn the same revenue, as shown formally in the following result.

Lemma 1 *The revenue earned by an* RA *is identical to its* TA.

Proof: Note that in an RA, the net payment received by the seller s comes directly from the nodes in the first level of the tree. The offset is zero, and the payment is calculated based on the maximum valuation in the subtree of the agents in the first level. The rest of the payments in the tree are internally adjusted within the nodes and does not reach the seller. Therefore, the total revenue earned by an RA can be simulated by transforming every first-level nodes with their valuations replaced with the maximum valuation of their subtree and applying $(\mathscr{g}, \mathscr{p})$ on those nodes. Hence, we have the lemma. ■

Given the above lemma, we can, WLOG, look only at the TAs for revenue maximization. In the TA of a given RA, the revenue maximization problem is restricted to the first level of the tree. However, the nodes of this restricted tree are the *transformed nodes* whose valuations are the maximum valuations of their respective subtrees. For notational simplicity, we use a fresh index ℓ to denote these transformed nodes at the first level, i.e., for $\texttt{children}(s)$. The *transformed valuation* of ℓ is denoted by $v_\ell := \max_{j \in \hat{T}_\ell} v_j$. Again, to reduce notational complexity, the set of the players in this TA is represented by $\tilde{N} := \texttt{children}(s)$. In the following, we state the fact that the v_ℓ's also follow the MHR property.

Fact 1 *The* maximum *of a finite number of i.i.d. random variables, each of which satisfies the MHR condition, also satisfies the MHR condition.*

Proof: Suppose, there are n i.i.d. random variables given by $X_1, X_2, \dots, X_n$, and their distribution is given by F. Let $Y := \max\{X_1, X_2, \dots, X_n\}$. It is given that F satisfies MHR condition, i.e., $f(x)/(1 - F(x))$ is monotone non-decreasing. Denote the distribution and density of Y by $\tilde{F}$ and $\tilde{f}$ respectively. Now,

$$\begin{aligned}\tilde{F}(x) &= P(Y \leqslant x) = P(\max\{X_1, X_2, \dots, X_n\} \leqslant x) \\ &= P(\cap_{i=1}^n \{X_i \leqslant x\}) = \prod_{i=1}^n P(X_i \leqslant x) = F^n(x).\end{aligned}$$

$$\text{Hence, } \tilde{f}(x) = nF^{n-1}(x) f(x).$$

$$\begin{aligned}\text{Therefore, } \frac{\tilde{f}(x)}{1 - \tilde{F}(x)} &= \frac{nF^{n-1}(x) f(x)}{1 - F^n(x)} \\ &= \frac{f(x)}{1 - F(x)} \cdot \left(\frac{n}{1 + \frac{1}{F(x)} + \frac{1}{F^2(x)} + \dots + \frac{1}{F^{n-1}(x)}} \right).\end{aligned}$$

Since $F(x)$ is non-decreasing, $1/F(x)$ is non-increasing. Hence, the denominator of the last term in the last expression is also non-increasing, leading the expression to be non-decreasing. Since $\frac{f(x)}{1-F(x)}$ is non-decreasing as well, we conclude that $\frac{\tilde{f}(x)}{1-\tilde{F}(x)}$ is also non-decreasing and hence Y satisfies MHR. ■

We now focus on the revenue maximization problem. Note that, $\mathscr{g}$ and $\mathscr{p}$ are particular choices of the allocations g and p respectively. Therefore, the expected allocations and payments are given by Eqns. (14) and (15) with g and p replaced with $\mathscr{g}$ and $\mathscr{p}$ respectively. In particular, the VIPC term in Eqn. (15) is zero for the nodes in the TA since the payment $\mathcal{p}$ sets it to zero for the nodes in the first level of the class RA.[6] Also, in the TA, the offset is zero. Therefore, $\rho_\ell = v_\ell, \; \forall \ell \in \tilde{N}$. The neighbor component of the types r_ℓ are no longer relevant since the mechanism is restricted to the first level in the TA. Hence, we can reduce the arguments of $\texttt{pay}_\ell$ and α_ℓ to only v_ℓ in Eqns. (14) and (15). Since, the only variable parameter in the payment of the agents is the allocation function $\mathcal{g}$, the optimization problem for revenue maximization in the RA class is given by

$$\begin{aligned} \max \quad & \sum_{\ell \in \tilde{N}} \int_{v_\ell = 0}^{b_\ell} \texttt{pay}_\ell(v_\ell) f_\ell(v_\ell) \, \mathrm{d}v_\ell \\ \text{s.t.} \quad & \mathcal{g} \text{ is monotone non-decreasing and deterministic} \end{aligned} \tag{19}$$

In the above equation, f_ℓ is the density of v_ℓ, which is assumed to have a bounded support of $[0, b_\ell]$. We will denote the corresponding distribution with F_ℓ. This optimization problem now reduces to the classic single item auction setting of Myerson [22]. Following that analysis, we find that the individual terms in the sum of the objective function of Eqn. (19) can be written as follows

$$\begin{aligned} & \int_{v_\ell = 0}^{b_\ell} \texttt{pay}_\ell(v_\ell) f_\ell(v_\ell) \, \mathrm{d}v_\ell \\ & = \int_0^{b_\ell} w_\ell(v_\ell) \alpha_\ell(v_\ell) f_\ell(v_\ell) \, \mathrm{d}v_\ell \\ & = \int_0^{b_\ell} w_\ell(v_\ell) \left(\int_{v_{-\ell}} \mathcal{g}_\ell(v_\ell, v_{-\ell}) f_{-\ell}(v_{-\ell}) \, \mathrm{d}v_{-\ell} \right) f_\ell(v_\ell) \, \mathrm{d}v_\ell \\ & = \int_v w_\ell(v_\ell) \mathcal{g}_\ell(v_\ell, v_{-\ell}) f(v) \, \mathrm{d}v. \end{aligned}$$

The expression $w_\ell(x) := x - (1 - F_\ell(x))/f_\ell(x)$ is defined as the *virtual valuation* of agent ℓ and for completeness, the derivation of the first equality is provided in the appendix. The second equality holds after expanding $\alpha_\ell(v_\ell)$ from Eqn. (14). The last equality holds since the valuations are independent (but may not be identically distributed as the number of nodes in the subtree of ℓ can be different from that of ℓ'), and f denotes the joint probability density of $(v_\ell, v_{-\ell})$.

The objective function of Eqn. (19) can therefore be written as

$$\int_v \left(\sum_{\ell \in \tilde{N}} w_\ell(v_\ell) \mathcal{g}_\ell(v_\ell, v_{-\ell}) \right) f(v) \, \mathrm{d}v.$$

The solution to the unconstrained version of the optimization problem given by Eqn. (19) is rather simple.

$$\begin{aligned} & \text{if } w_\ell(v_\ell) < 0, \forall \ell \in \tilde{N}, \text{ then } \mathcal{g}_\ell(v_\ell, v_{-\ell}) = 0, \forall \ell \in \tilde{N} \\ & \text{else } \mathcal{g}_\ell(v_\ell, v_{-\ell}) = \begin{cases} 1 & \text{if } w_\ell(v_\ell) \geqslant w_k(v_k), \forall k \in \tilde{N} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{20}$$

The ties in $w_\ell(v_\ell)$ are broken arbitrarily. Since the distributions of $v_\ell, \ell \in \tilde{N}$ satisfy MHR, the virtual valuations, w_ℓ, are monotone non-decreasing. Also, since this mechanism breaks the tie arbitrarily in favor of an agent, the allocation is also deterministic. Therefore, the optimal solution of the unconstrained problem of Eqn. (19) also happens to be the optimal solution of the constrained problem. We find the payments of the winner from Eqn. (18) as follows.

$$\begin{aligned} \text{define } \kappa_\ell^*(v_{-\ell}) &= \inf\{y : \mathcal{g}_\ell(y, v_{-\ell}) = 1\}, \\ \mathcal{p}_\ell(v_\ell, v_{-\ell}) &= \kappa_\ell^*(v_{-\ell}) \cdot \mathcal{g}_\ell(v_\ell, v_{-\ell}), \end{aligned} \tag{21}$$

where $\kappa_\ell^*(v_{-\ell})$ is the minimum valuation of agent ℓ to become the winner. Formally, we define the auction as follows.

[6]The VIPC term needs to be non-positive for the auction to be IIR, and since our objective is to maximize revenue, it must be zero. This is ensured by $\mathcal{p}_i$.

Definition 10 (Maximum Virtual Valuation Auction (`maxViVa`)) *The* maximum virtual valuation *auction is a subclass of* `RA`*, where the* `TA`*s of that subclass follow the allocation and payments given by Eqns.* (20) *and* (21) *respectively.*

We consolidate the arguments above in the form of the following theorem.

Theorem 9 *For agents having i.i.d. MHR valuations, the revenue-optimal* `RA` *is* `maxViVa`.

Since multiple `RA`s can reduce to the same `TA`, the revenue optimal `RA` is a class of auctions, all belonging to `RA`, that has the same `TA` given by Def. 10. Note that neither IDM nor `LbLEV` is `maxViVa` because they do not use any priors. Therefore, the revenue-maximizing auctions in this setting are a new class of mechanisms that have not been explored in the literature.

6.4 Extension to non-i.i.d. agents and any diffusion auction

When we migrate away from i.i.d. valuations, it is not clear if the nodes in the `TA` satisfy MHR or a relatively weaker condition of *regularity* (which only requires the virtual valuations to be non-decreasing). Hence, the revenue maximization problem becomes far more challenging.

We provide an experimental study in the next section that shows that if the i.i.d. assumption does not hold, a special auction from the `LbLEV` class can yield more revenue than the currently known network auctions.

To generalize our results beyond `RA` for revenue maximization, we need to consider the revenue maximization problem Eqn. (19) with the constraints of `MFFE` (Def. 8). This optimization problem seems to have much less structure than that in `RA`. Therefore, we need more structural results about the $\texttt{pay}_i$ terms so that this optimization problem can be simplified. Also, we cannot work with only `TA`s anymore since the revenue-optimal `DA` may not be reducible to a `TA` (*à la* Lemma 1). We address these problems in our future work.

7 Designing `LbLEV` for Improving Revenue

In the previous section, we investigated the case of i.i.d. valuations. In this section, we show that if the valuations are non-i.i.d., there exists non-trivial diffusion auctions that yield higher expected revenue than the known diffusion auctions in literature.

`LbLEV` generates a class of mechanisms (Alg. 1) for different choices of the exponents $t \in \mathbb{R}^n_{>0}$ and trees $\hat{T}$. While each mechanism in that class is `DDSIC` and IR, we can anticipate that certain choices may lead to a higher revenue collected by the auction. In this section, we test that hypothesis and find out how the exponents t can be designed such that it improves the revenue over the other `DDSIC` and IR auctions known in the literature. The revenues are compared for the same tree $\hat{T}$ for all competing auctions.

Revenue maximization in an auction is typically done in a prior-based approach [12, 21, 22]. Such approaches assume that the distribution of the types is known by the mechanism designer from its past interactions with the agents. We adopt a similar prior-based approach. In the context of network auctions, the types of the agents consist of the valuations and their neighbors' set and we assume a prior on both.

Other `DDSIC` mechanisms to compare. Guo and Hao [11] provide a comprehensive survey of the auctions on the network. TNM and CDM reduce to IDM when restricted to a tree. The objective of FDM and NRM is to redistribute the revenue so that there is little surplus – hence, they are unsuitable for a revenue comparison. Other mechanisms like MLDM, CSM, and WDM apply to very specific settings, e.g., distribution markets with intermediatries, economic networks, and weighted networks respectively. Hence, we find that IDM is the sole candidate to compare with `LbLEV`.

Two-stage tree generation model. For the experiments, we need to iterate over randomly generated trees connecting the agents, and also equip the mechanism designer with some prior information about the connections model. To do so, we adopt a two-stage tree generation model, where the first stage is observable by the designer, but the second stage is not. It resembles the situation: *the designer knows who can probably be the children of which nodes, but cannot deterministically observe it while designing the mechanism.*

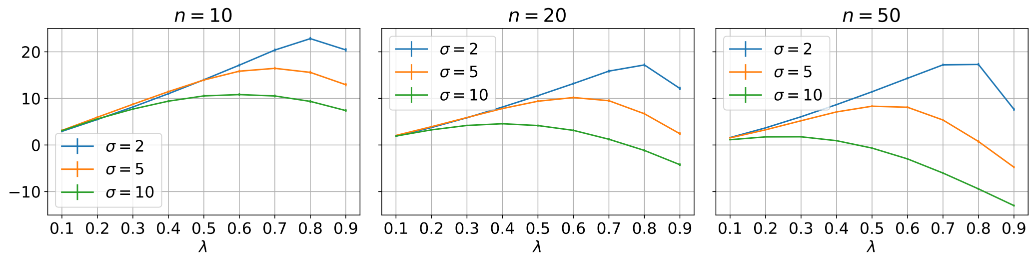

Figure 4: Percentage increase in revenue for `LbLEV` over IDM w.r.t. λ for different σ and n.

The mechanism `LbLEV` extracts a tree $\hat{T}$ from the graph generated through the reported r_i's. For our experiments, we create the r_i's in two stages. First, we generate a *base* tree in the following way. We fix the number of nodes n in this tree. Starting from the seller, which is not a part of the agent set, a random *children set* is picked for each node at every level of the tree where the size of the children set is drawn uniformly at random between 1 and $\lfloor n/3 \rfloor$ from the rest of agents without replacement. This process is continued until all the n nodes are exhausted (the last parent node gets the remaining number of nodes when it draws more than that remaining number).

The second stage probabilistically *activates* the edges on this base tree. The activated set of edges provides a sub-tree of the base tree, and this is considered to be the final tree $\hat{T}$. More concretely, once the base tree is realized, the actual set r_i is generated by tossing a coin with probability drawn from Beta$(5, 1)$ for each edge from i to its children. We assume that the second stage of randomization is not part of the prior information available to the mechanism designer, and hence, the choice of the exponent vector t cannot depend on it. The second stage, therefore, helps us to cross-validate the mechanism designed from the prior information.

Valuation generation model. The valuations are drawn independently from $\mathcal{N}(\mu, \sigma^2)$. We assume that there are *three* classes of agents: `high`, `medium`, and `low`, having μ to be 100, 70, and 50 respectively, and the same σ. We will see the effect of σ on the revenue in our experiments.

The prior information for the designer consists of the first stage of the tree generation process and the valuation generation process.

Setting the exponent vector t. Suppose, we knew that the nodes i^* and ℓ are the first level nodes whose subtrees are the `winner` and `runnerup` respectively in `LbLEV`. From Alg. 1, we know that the effective valuations ρ_i of a first level node i in the tree $\hat{T}$ is the maximum valuation of the nodes in the subtree of i, i.e., $\hat{T}_i$. Note that, the revenue generated in `LbLEV` is $\rho_\ell^{t_\ell/t_{i^*}}$. Hence, a larger exponent ratio t_ℓ/t_{i^*} yields a better revenue as long as $\rho_\ell^{t_\ell/t_{i^*}} \leqslant \rho_{i^*}$. This is the driving philosophy of the following choice of t with only the prior information.

On the base tree, we replace the nodes' valuations with the means, which is a prior information, and call the node having the highest and second highest ρ_i's in the first level of the base tree to be the *expected* `winner` and `runnerup` respectively. Suppose, these two mean valuations are $w^e_{\texttt{winner}}$ and $w^e_{\texttt{runnerup}}$ respectively. We set the exponent of the first level *expected* `runnerup` is set to $(1 - \lambda) \cdot 1 + \lambda \cdot \frac{\log w^e_{\texttt{winner}}}{\log w^e_{\texttt{runnerup}}}$, with λ being a parameter chosen by the mechanism. This is a convex combination between 1, which is the exponent ratio for IDM, and the other extreme $\frac{\log w^e_{\texttt{winner}}}{\log w^e_{\texttt{runnerup}}}$. If the true `winner` and the `runnerup` would have indeed come from the subtrees of the expected `winner` and `runnerup`, then the exponent ratio $\frac{\log v_{\texttt{winner}}}{\log v_{\texttt{runnerup}}}$ would have extracted the maximum revenue in `LbLEV`. This is the intuition of using this factor as a candidate for the expected `runnerup`'s exponent.

The exponents of all other agents (including the expected `winner`) are set to 1. Note that t is decided based on the first stage of the tree generation process and the prior of the valuation. It is independent of the second level of the tree generation process, and therefore, is agnostic of the actual tree $\hat{T}$. Such a t is indepedent of the agents' actions and is consistent with Alg. 1. Indeed there is a possibility that a probabilistic draw of the second stage of the tree generation process may have a different `winner` and `runnerup` than their expected ones, which makes this choice of t sub-optimal than IDM for revenue.

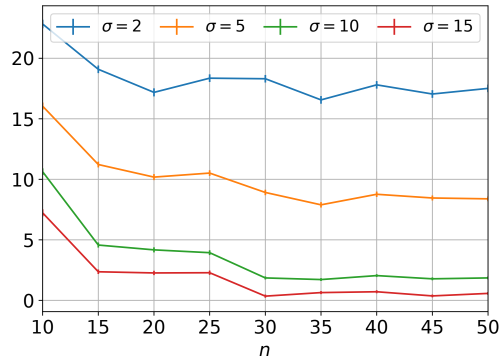

Figure 5: Percentage increase in revenue for `LbLEV` over IDM for different σ and *learned* λ^* (and hence t).

First set of experiments. In this set of experiments, we find the effect of the three parameters, n, σ, and λ, on the revenue of the two auctions: (1) `LbLEV` with the chosen exponent vector t as above and (2) IDM. We consider one agent from class `high`, $\lfloor(n-1)/2\rfloor$ agents from class `medium`, and $\lceil(n-1)/2\rceil$ agents from class `low`. This choice is to observe how the exponents t make a difference in the revenue earned. If there are many agents of class `high`, then it is highly probable that both the expected `winner` and the `runnerup` in the first level of the base tree has the same mean valuation, which makes the optimal exponent to be unity – same as IDM. In the first experiment, we consider different values for σ and n, and compare the revenue earned by `LbLEV` and IDM with varying λ. The results are shown in Fig. 4. For every λ, the base graph and the edge-activation probability (drawn from Beta(5,1)) generation have been repeated 100 times, and for each of such instances the edge activation and valuation generation for all agents have been repeated 100 times. The plot shows the mean percentage improvement of the revenue of `LbLEV` over IDM, with the standard error around it. Observe that, for every pair of σ and n, there is an optimal convex combination (say λ^*) for which the revenue gap between `LbLEV` and IDM reaches a maxima. Recall that this λ^* also determines the exponent of the expected `runnerup` in the base tree.

Second set of experiments. The first set of experiments gives us the insight that the optimal convex combination factor λ^* depends on n and σ. Motivated by this observation, in this set of experiments, we run a regression model to learn the optimal $\lambda^*(n, \sigma)$ from several such instances of (n, σ, λ^*). We used the *random forest regressor* [3] with the parameter of number of decision trees set to 100 to learn the λ^* function. We chose the random forest regressor for two reasons: (a) from the examples, the function seems non-linear and instead of choosing a fixed non-linear function, an ensemble regressor could perform better, and (b) random forest gave the best performance among the few other ensemble regressors we tested with (e.g., `ADABOOST`, `GradientBoosting`). We find that with the learned λ^*, which yields the exponents t, `LbLEV` performs better than IDM. For certain choices of (σ, n), particularly when both n and σ are large, the learned exponents are close to 1 for all agents, almost reducing `LbLEV` to IDM. Fig. 5 shows the results.

8 Summary and Plans of Extension

In this paper, we provided a characterization of randomized truthful single indivisible item auctions on a network. Our results are the network counterpart of Myerson's result [22]. We obtained a detailed description of the revenue optimal mechanism for a class called referral auctions with i.i.d. MHR valuations. When i.i.d. assumption does not hold, we provided a mechanism from our characterized class of `DDSIC` and IR auctions to experimentally show an improvement in the revenue from the currently known diffusion mechanisms. The question of finding the revenue optimal mechanism for a general network is still open and we want to pursue that as a future work.

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References

[1] Moshe Babaioff, Shahar Dobzinski, Sigal Oren, and Aviv Zohar. On Bitcoin and Red Balloons. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC 2012)*, pages 56–73. ACM, 2012.

[2] Richard E Barlow, Albert W Marshall, and Frank Proschan. Properties of probability distributions with monotone hazard rate. *The Annals of Mathematical Statistics*, pages 375–389, 1963.

[3] Leo Breiman. Random forests. *Machine learning*, 45(1):5–32, 2001.

[4] E. H. Clarke. Multipart pricing of public goods. *Public Choice*, 11:17–33, 1971.

[5] Sven De Vries and Rakesh V Vohra. Combinatorial auctions: A survey. *INFORMS Journal on computing*, 15(3):284–309, 2003.

[6] Marguerite DeLiema, Doug Shadel, Amy Nofziger, and Karla Pak. Aarp study of multilevel marketing: Profiling participants and their experiences in direct sales. *AARP Foundation*, 2018.

[7] Fabio Drucker and Lisa Fleischer. Simpler Sybil-Proof Mechanisms for Multi-Level Marketing. In *Proceedings of ACM Electronic Commerce (EC 2012)*, pages 441–458. ACM, 2012.

[8] Yuval Emek, Ron Karidi, Moshe Tennenholtz, and Aviv Zohar. Mechanisms for Multi-Level Marketing. In *Proceedings of the 12th ACM Conference on Electronic Commerce (EC 2011)*, pages 209–218. ACM, 2011.

[9] J. R. Green and J.-J. Laffont. *Incentives in Public Decision Making*. North-Holland, 1979.

[10] T. Groves. Incentives in teams. *Econometrica*, 41:617–631, 1973.

[11] Yuhang Guo and Dong Hao. Emerging methods of auction design in social networks. In *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI-21*, pages 4434–4441, 8 2021. Survey Track.

[12] Sergiu Hart and Noam Nisan. Approximate revenue maximization with multiple items. *Journal of Economic Theory*, 172:313–347, 2017.

[13] John H Kagel and Dan Levin. Auctions: A survey of experimental research, 1995-2010. *Handbook of experimental economics*, 2:563–637, 2011.

[14] Joosung Lee. Mechanisms with referrals: Vcg mechanisms and multilevel mechanisms. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 789–790, 2016.

[15] Bin Li, Dong Hao, Dengji Zhao, and Tao Zhou. Mechanism design in social networks. In *Thirty-First AAAI Conference on Artificial Intelligence*, 2017.

[16] Bin Li, Dong Hao, Dengji Zhao, and Tao Zhou. Customer sharing in economic networks with costs. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence*, pages 368–374, 2018.

[17] Bin Li, Dong Hao, Dengji Zhao, and Makoto Yokoo. Diffusion and auction on graphs. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence*, pages 435–441, 2019.

[18] Bin Li, Dong Hao, Fan Liu, Hui Gao, and Tao Zhou. Information diffusion and revenue optimization in distribution market. *IEEE Access*, 8:40850–40860, 2020.

[19] Bin Li, Dong Hao, and Dengji Zhao. Incentive-compatible diffusion auctions. In *Proceedings of the Twenty-Ninth International Conference on International Joint Conferences on Artificial Intelligence*, pages 231–237, 2020.

[20] Bin Li, Dong Hao, Hui Gao, and Dengji Zhao. Diffusion auction design. *Artificial Intelligence*, 303:103631, 2022.

[21] Alejandro M Manelli and Daniel R Vincent. Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *Journal of Economic theory*, 137(1):153–185, 2007.

[22] Roger B. Myerson. Optimal Auction Design. *Mathematics of Operations Research*, 6:1767–1797, 1981.

[23] Peter J Vander Nat and William W Keep. Marketing fraud: An approach for differentiating multilevel marketing from pyramid schemes. *Journal of Public Policy & Marketing*, 21(1): 139–151, 2002.

[24] Swaprava Nath and Arunava Sen. Affine maximizers in domains with selfish valuations. *ACM Transactions on Economics and Computation (TEAC)*, 3(4):1–19, 2015.

[25] Yaniv Reingewertz. An economic model of multi-level marketing. *Plos one*, 16(7):e0253700, 2021.

[26] R.T. Rockafellar. *Convex Analysis: (PMS-28)*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 2015. ISBN 9781400873173. URL `https://books.google.co.in/books?id=jzpzBwAAQBAJ`.

[27] Y. Shoham and K. Leyton-Brown. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press, 2008.

[28] W. Vickrey. Counter speculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16(1):8–37, 1961.

[29] Wen Zhang, Dengji Zhao, and Hanyu Chen. Redistribution mechanism on networks. In *Proceedings of the 19th International Conference on Autonomous Agents and Multi-Agent Systems*, pages 1620–1628, 2020.

[30] Wen Zhang, Dengji Zhao, and Yao Zhang. Incentivize diffusion with fair rewards. In *ECAI 2020*, pages 251–258, 2020.

Appendix

A Derivation of the virtual valuation

We need to show that

$$\int_0^{b_\ell} \texttt{pay}_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell = \int_0^{b_\ell} w_\ell(v_\ell)\alpha_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell, \tag{22}$$

where,

$$\texttt{pay}_\ell(v_\ell) = v_\ell \alpha_\ell(v_\ell) - \int_0^{v_\ell} \alpha_\ell(y)\,\mathrm{d}y, \text{ and,} \tag{23}$$

$$w_\ell(v_\ell) = v_\ell - \frac{1 - F_\ell(v_\ell)}{f_\ell(v_\ell)}. \tag{24}$$

Substituting Eqn. (23) in the LHS of Eqn. (22), we get

$$\begin{aligned}
&\int_0^{b_\ell} \texttt{pay}_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell - \int_0^{b_\ell}\int_0^{v_\ell} \alpha_\ell(y) f_\ell(v_\ell)\,\mathrm{d}y\,\mathrm{d}v_\ell \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell - \int_0^{b_\ell}\int_y^{b_\ell} \alpha_\ell(y) f_\ell(v_\ell)\,\mathrm{d}v_\ell\,\mathrm{d}y \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell - \int_0^{b_\ell} \alpha_\ell(y) \left(\int_y^{b_\ell} f_\ell(v_\ell)\,\mathrm{d}v_\ell\right)\mathrm{d}y \\
&= \int_0^{b_\ell} v_\ell \alpha_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell - \int_0^{b_\ell} \alpha_\ell(y)(1 - F_\ell(y))\,\mathrm{d}y \\
&= \int_0^{b_\ell} \left(v_\ell - \frac{1 - F_\ell(v_\ell)}{f_\ell(v_\ell)}\right) \alpha_\ell(v_\ell) f_\ell(v_\ell)\,\mathrm{d}v_\ell.
\end{aligned}$$

In the second equality, we interchange the order of integration. The integrable space has the following order: y varying from $0 \rightarrow v_\ell$ and thereafter v_ℓ varying from $0 \rightarrow b_\ell$ which is equivalent to v_ℓ varying from $y \rightarrow b_\ell$ and thereafter y varying from $0 \rightarrow b_\ell$. The third equality holds by taking the $\alpha_\ell(y)$ term outside the inner integral since it is independent of v_ℓ. The next equality holds since the distribution of v_ℓ has the support of $[0, b_\ell]$, hence at b_ℓ, the value of F_ℓ is unity. The last equality is obtained by using the same integration varible for both integrals and rearranging them.