A Gale-Shapley View of *Unique* Stable Marriages

Kartik Gokhale[1], Amit Kumar Mallik[1], Ankit Kumar Misra[1], and Swaprava Nath[1]

[1]Indian Institute of Technology Bombay, Mumbai, India,
kartikgokhale2000@gmail.com,19D070007@iitb.ac.in,,
ankitkmisra@ucla.edu,swaprava@iitb.ac.in

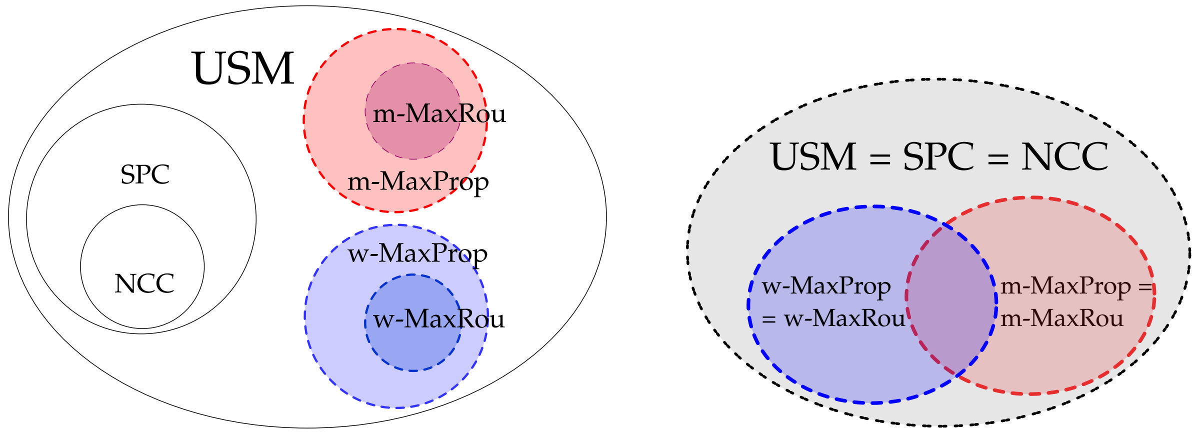
Abstract

Stable marriage of a two-sided market with unit demand is a classic problem that arises in many real-world scenarios. In addition, a unique stable marriage in this market simplifies a host of downstream desiderata. In this paper, we explore a *new* set of sufficient conditions for *unique stable matching* (USM) under this setup. Unlike other approaches that also address this question using the structure of preference profiles, we use an algorithmic viewpoint and investigate if this question can be answered using the lens of the *deferred acceptance* (DA) algorithm (Gale and Shapley, 1962). Our results yield a set of sufficient conditions for USM (viz., MaxProp and MaxRou) and show that these are disjoint from the previously known sufficiency conditions like *sequential preference* and *no crossing*. We also provide a characterization of MaxProp that makes it efficiently verifiable, and shows the gap between MaxProp and the entire USM class. These results give a more detailed view of the sub-structures of the USM class.

1 Introduction

The *stable marriage problem* considers a two-sided market where agents of each side (e.g., men) is assumed to have a linear preference over the other side (e.g., women) and matches are one-to-one, i.e., each agent has a single demand. Stability asks for a pairing between these agents such that there does not exist any pair of a man and a woman who would like to abandon the current matching and mutually prefer a marriage among themselves. Gale and Shapley (1962) proved that such a stable matching always exists and is obtained via a computationally simple algorithm called *deferred acceptance* (DA). However, there could be multiple stable matchings and it raises questions on which one to pick. The stable matching problem is very well studied in the literature and several useful results exists related to DA and its variants. For instance, the questions regarding the maximum (Karlin *et al.*, 2018) or average number of stable matchings (Pittel, 1989), complexity of counting stable marriages (Irving and Leather, 1986), matching with incomplete lists (Iwama *et al.*, 2002), indifferences (Manlove, 2002), heterogeneous jobs and workers (Crawford and Knoer, 1981), and many more, have already been investigated. See Iwama and Miyazaki (2008) for a comprehensive survey on the stable matching problem and Roth (2008) for a survey of the DA-type algorithms.

In this context, uniqueness of stable matching (Eeckhout, 2000; Clark, 2006) has a very important place. First, since the actual pairings of men and women are a stable matching based on their *reported* preferences, a normative goal is to ensure that it is indeed their *actual* preferences, i.e., the stable matching algorithm is *strategyproof*. However, it is known that DA is not strategyproof for a non-proposer (Gale and Sotomayor, 1985a) unless there is a unique stable matching. Though a unique stable matching is not sufficient for strategyproofness (Roth, 1989) except in the incomplete information setup (Ehlers and Massó, 2007), it is a



(a) $n \geqslant 3$ **(b)** $n = 2$

Figure 1: *The above two figures show the sub-structures of the* USM *class for* $n \geqslant 3$ *and* $n = 2$ *respectively. The dashed lines and the shaded regions denote the new sub-structures of* USM *that are the contributions of this paper. We also characterize the class* MaxProp *and its gap with* USM*. In Figure 1b, the fact* USM = SPC *was known from Eeckhout (2000). However, we provide a more direct proof of this fact.*

property from which further structures of strategyproofness can be obtained. We define the class of preference profiles where the set of stable matchings is a singleton as *unique stable matching* (USM) in this paper.

The second reason why USM is desirable is the anti-symmetry of the preferences of men and women over the stable matchings. It is known that between two different stable matchings μ_1 and μ_2, if μ_1 is at least as preferred as μ_2 by all men, then μ_2 must be at least as preferred as μ_1 by all women, i.e., men and women have exactly opposite preferences over the stable matchings (Gale and Sotomayor, 1985b). Hence, finding a stable matching that is *unbiased* to any side of the market is often challenging. A considerable amount of research effort has been put to find a *fair* compromise between the two extremes (see, e.g., Klaus and Klijn (2006); Tziavelis *et al.* (2020); Brilliantova and Hosseini (2022)). However, the question of bias also does not appear in the USM class since there is exactly one stable matching.

Finally, unique stable matchings have appeared in many real-world matching markets, e.g., in the US National Resident Matching Program (Roth and Peranson, 1999), Boston school choice (Pathak and Sönmez, 2008), online dating (Hitsch *et al.*, 2010), and the Indian marriage market (Banerjee *et al.*, 2013).

In this paper, we aim to understand the internal structure of the USM using a DA algorithmic lens.

1.1 Our contributions

The main contributions of this paper are as follows (illustrated graphically in Figure 1).

- We view the USM problem using the number of proposals and rounds in the classic Gale-Shapley DA-algorithm, and introduce two new conditions m-MaxProp and m-MaxRou (similarly w-MaxProp and w-MaxRou), defined w.r.t. men(women)-proposing DA. We show the mutual relationship of these two properties in Theorem 2 when the number of men(or women) $|M|(= |W|) = n \geqslant 3$. We show that each of these conditions is *sufficient* for USM (Theorem 3).

- The most prominent existing sufficient conditions for USM, the *sequential preference con-*

dition (SPC (Eeckhout, 2000)) and the *no crossing condition* (NCC (Clark, 2006)), are disjoint from the new sufficient conditions proposed in this paper for $n \geqslant 3$ (Theorem 4). Hence, it makes the internal sub-structure of the USM class outside NCC and SPC clearer. However, under this scenario, the men and women proposing versions of MaxProp class turns out to be disjoint as well (Theorem 5).

- When $n = 2$, we show that the classes m-MaxProp and m-MaxRou (similarly w-MaxProp and w-MaxRou) coincide (Theorem 6) and so do SPC and NCC (Theorem 7). Also, m-MaxProp and w-MaxProp are contained within SPC for $n = 2$ (Theorem 8). We also provide a direct proof of the fact that for $n = 2$, USM and SPC are equivalent (Theorem 9), a result originally proved by Eeckhout (2000). However, we also point out an inconsistency in the claim of SPC being necessary for USM for $n = 3$ (Eeckhout, 2000) through Example 4.

- Interestingly, for $n = 2$, the classes m-MaxProp and w-MaxProp have an overlap and we characterize it in Theorem 10.

- We characterize the class MaxProp in Theorem 11 and these characterizing conditions are efficiently verifiable. This result also shows the gap between the two classes: USM and MaxProp (applies to both versions of MaxProp).

1.2 Related works

Several works focus on finding sufficient conditions for USM, e.g., the sequential preference condition (Eeckhout, 2000), the no crossing condition (Clark, 2006), the co-ranking condition (Legros and Newman, 2010), the acyclicity condition (Romero-Medina and Triossi, 2013), the universality condition (Holzman and Samet, 2014), oriented preferences (Reny, 2021), and aligned preferences (Niederle and Yariv, 2009). These results provide structural views of the preference profiles that lead to uniqueness in the stable matchings. Finding a necessary condition has also been investigated and there are two prominent approach techniques. The first one uses an idea of α-reducibility, proposed originally by Alcalde (1994). A marriage problem satisfies α-reducibility if every sub-population of men and women has a *fixed pair* (a pair of man and woman who prefer each other the most). Clark (2006) shows that this condition is necessary as well for USM.

A different approach to this problem uses the idea of *acyclicity*, originally proposed by Chung (2000). Acyclicity implies that if the agents point to their most preferred partners, then the resulting directed graph should not have any directed cycle. While Romero-Medina and Triossi (2013) show that it is a sufficient condition for USM, the necessity condition using this method is explored recently by Gutin *et al.* (2021). Gutin *et al.* (2021) use the acyclicity on a reduced graph that they define as the *normal form*. The idea of normal form is used for submatching markets by Irving and Leather (1986), and Balinski and Ratier (1997). Gutin *et al.* (2021) claim that the difficulty in finding a necessary condition for USM in these approaches was that the acyclicity property was being used on the complete preference profile, while the entire preference profile may not be relevant for a unique stable match. Using the idea of normal form, they prune the preferences where an agent can never match with certain partners in any stable matching. This acyclicity on a normal form turns out to be necessary and sufficient for USM (Gutin *et al.*, 2021).

Our approach differs considerably in the way our conditions are defined. Instead of looking at the USM class through the preference structures of the players, we view it using the DA algorithm and its execution over a profile. Our results consider the maximum number of proposals made by the agents and the number of rounds in DA, and provides the extra structures that yields a clearer view of the space between the currently known sufficient conditions and the USM class (Figure 1). It shows that indeed an algorithm can also help clarify the structure of USM.

2 Preliminaries

Consider a two-sided unit-demand matching market, where the two sides are represented, WLOG, by men and women respectively. The agents of each side are expressed as two equi-cardinal finite sets, denoted by M and W, $|M| = |W| = n$, respectively. The sets share no common agents, i.e., $M \cap W = \emptyset$. All men have strict preferences over all women and vice versa. Individual preferences, denoted $\succ_i$ for agent i, are assumed to be complete, transitive, and anti-symmetric. The notation $m_i \succ_{w_k} m_j$ denotes $w_k \in W$ prefers $m_i \in M$ over $m_j \in M$, and similarly, $w_i \succ_{m_k} w_j$ denotes $m_k \in M$ prefers $w_i \in W$ over $w_j \in W$. The preference profile is denoted by $\succ := \{\succ_i : i \in M \cup W\}$. The set of all complete, transitive, and anti-symmetric preference profiles in this setup is denoted by $\mathcal{P}$. A *matching* and several other definitions in this setting follow Gale and Shapley (1962).

Definition 1 (Matching). A matching in $\succ$ is a mapping μ from $M \cup W$ to itself such that for every man $m \in M, \mu(m) \in W$, for every woman $w \in W, \mu(w) \in M$, and for every $m, w \in M \cup W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

The above definition says that each man is matched to exactly one woman and vice-versa. To define stability of a matching, we need the definition of *blocking pair* as given below.

Definition 2 (Blocking Pair). A pair $(m, w), m \in M, w \in W$ is a blocking pair of a matching μ in $\succ$ if $m \succ_w \mu(w)$ and $w \succ_m \mu(m)$.

Informally, the above definition means that the pair (m, w) prefer each other over each of their currently matched partners. This leads to the definition of stable matching as follows.

Definition 3 (Stable Matching). A matching μ in $\succ$ is *stable* if it does not have any blocking pair.

Gale and Shapley (1962) showed that for any preference profile $\succ$, a stable matching always exists and can be found via the *deferred acceptance* (DA) algorithm. The working principle of this algorithm is the following. The algorithm comes in two versions based on whether the men or the women are the proposers. In every round of the men-proposing DA algorithm, each unmatched man proposes his favorite woman that has not rejected him already. The women, in that round, receive the proposals and *tentatively* accepts the most favorite man that has proposed to her and rejects the rest. The rejected men go to the next round and repeat this activity. The algorithm stops when no man is rejected in a round. A formal representation is given in Algorithm 1.

Though the algorithm always converges to a stable matching, it is also known that the men-proposing DA and the women-proposing DA converges to men and women optimal stable matchings respectively, which could be quite different. There is a hierarchy among the stable matchings from the men and women points of view as given by the following result.

Theorem 1 (Gale and Sotomayor (1985b))**.** *If for any two distinct stable matchings μ_1 and μ_2 in $\succ$, if each man find μ_1 at least as preferred as μ_2, then every woman will find μ_2 at least as preferred as μ_1.*

The subclass of $\mathcal{P}$ where the set of stable matchings is a singleton is defined as the *unique stable matching* (USM) class. In USM, the men and women proposing DA reaches the same stable matching. Because of the various satisfactory properties exhibited by this class as discussed in Section 1, there had been various attempts to characterize the structures of the

Algorithm 1: (Men-proposing) Deferred Acceptance (DA)

```
Parameters : M = {m_1, ..., m_n}, W = {w_1, ..., w_n}, ≻ = {≻_i : i ∈ M ∪ W}
for i ∈ M ∪ W do
    μ(i) ← ∅
while ∃m ∈ M such that μ(m) = ∅ do
    w ← highest woman in ≻_m to whom m has not proposed yet
    if ∃m' ∈ M such that μ(m') = w and μ(w) = m' then
        if m ≻_w m' then
            μ(m) ← w, μ(w) ← m
            μ(m') ← ∅
    else
        μ(m) ← w, μ(w) ← m
return μ
```

preference profiles in USM. In the following section, we introduce two prominent sufficient conditions for USM.

Remark. There are necessity results of USM as well, using ideas like α-reducibility (Clark, 2006) and *acyclicity* using a *normal form* of the preferences (Gutin *et al.*, 2021). However, in this paper, our objective is to view the USM class from a DA algorithmic perspective and we discuss how our results can be applicable even in domains with partial preferences and in practical scenarios in Section 5.

3 Current State-of-the-art Sufficient Conditions

Though there has been various sufficient conditions proposed for USM, (Romero-Medina and Triossi, 2013; Gusfield and Irving, 1989; Reny, 2021, e.g.), the *sequential preference condition* (SPC, (Eeckhout, 2000)) and *no crossing condition* (NCC, (Clark, 2006)) provide a deeper structural view of the preference profiles of the agents that gives rise to USM.

Definition 4 (Sequential Preference Condition). A preference profile $\succ$ satisfies *sequential preference condition* (SPC) if there exists an ordering of men, $m_1, m_2, \ldots, m_n$, and women, $w_1, w_2, \ldots, w_n$, such that

1. man m_i prefers w_i over $w_{i+1}, w_{i+2}, \ldots, w_n$, and
2. woman w_i prefers m_i over $m_{i+1}, m_{i+2}, \ldots, m_n$.

Eeckhout (2000) showed that SPC is sufficient for uniqueness in the stable matching, however, not necessary for $n \geqslant 3$ as we show in the example below.

Example 1 (USM but not SPC)**.** Consider the following preference profile.

$$\begin{pmatrix} m_1: & w_2 \succ w_1 \succ w_3 & & w_1: & m_1 \succ m_2 \succ m_3 \\ m_2: & w_1 \succ w_2 \succ w_3 & ; & w_2: & m_2 \succ m_3 \succ m_1 \\ m_3: & w_1 \succ w_2 \succ w_3 & & w_3: & m_3 \succ m_2 \succ m_1 \end{pmatrix}$$

This is not SPC, since SPC needs at least one pair of man and woman that rank each other at the top. However, the men-proposing DA yields the matching where m_i is matched with w_i, $i = 1, 2, 3$, which is the men-optimal matching. However, in this case, that is the women-optimal as well since each woman gets her top preference. By Theorem 1, this profile has an unique stable matching, i.e., it belongs to USM. □

Later, Clark (2006) defined a refinement to this condition that implies SPC.

Definition 5 (No Crossing Condition). A preference profile $\succ$ satisfies *no crossing condition* (NCC) if there exists an ordering $(m_1, m_2, \ldots, m_n)$ of M and an ordering $(w_1, w_2, \ldots, w_n)$ of W, such that if $i < j$ and $k < l$, then

1. $w_l \succ_{m_i} w_k \Rightarrow w_l \succ_{m_j} w_k$, and
2. $m_j \succ_{w_k} m_i \Rightarrow m_j \succ_{w_l} m_i$.

This condition implies that if the men and women are lined up in that given order and any pair of men (or women) are asked to point to his (or her) favorite partner among a pair of potential partners, their pointers cannot cross each other. Though NCC implies SPC, the converse is not true for $n \geqslant 3$. The following example by Clark (2006) shows that a profile $\succ$ can satisfy SPC but not NCC.

Example 2 (SPC but not NCC)**.** Consider the following preference profile.

$$\begin{pmatrix} m_1 : & w_1 \succ w_2 \succ w_3 & & w_1 : & m_1 \succ m_2 \succ m_3 \\ m_2 : & w_2 \succ w_3 \succ w_1 & ; & w_2 : & m_1 \succ m_2 \succ m_3 \\ m_3 : & w_1 \succ w_2 \succ w_3 & & w_3 : & m_3 \succ m_2 \succ m_1 \end{pmatrix}$$

In this example, SPC is satisfied in the order $(m_1, m_2, m_3), (w_1, w_2, w_3)$. But it is not possible to order $\{m_2, m_3\}$ and $\{w_1, w_3\}$ (and therefore M and W) to satisfy the conditions of Definition 5 (i.e., to avoid a crossing). Thus, this profile does not satisfy NCC. □

These current sufficient conditions for $n \geqslant 3$ are shown on the LHS of Figure 1a. These sufficient conditions, however, become identical with USM for $n = 2$ and we discuss this in Section 4 in detail. Example 1, however, shows that there exists unexplored space outside the sufficient condition SPC that are USM. We provide additional structure to that space in this paper.

4 Our Results

This paper considers the USM problem from the DA perspective. We need to use the following result to define two new conditions that we later prove to be sufficient for USM. The definitions deal with the number of proposals a woman gets in a men-proposing DA and the number of rounds of proposals in DA. In the rest of the paper, WLOG, we use men-proposing DA whenever we consider DA. However, the same definitions and results hold for a symmetrically opposite women-proposing version as well.

Fact 1. *In a men-proposing* DA *algorithm, there exists a woman $w \in W$ who receives exactly one proposal.*

Proof. We prove this by contradiction. Suppose, there exists a preference profile $\succ$ where each woman gets at least two proposals. According to the DA algorithm, every woman in that case will accept exactly one of them and reject the rest of the proposals. Consider the last round of the DA algorithm. In this round, there must exist a woman who has more than one (not necessarily new) proposals. This holds since every woman is proposed to at least twice by assumption. But, by the algorithm she must reject at least one proposal, which contradicts that this is the last round. Hence the lemma is proved. □

Fact 2. *In the men-proposed* DA *algorithm*

1. *the maximum possible number of proposals is $n^2 - n + 1$, and*

2. *the maximum possible number of rounds is* $n^2 - 2n + 2$.

Both the bounds are achievable, *i.e., there exists a preference profile* $\succ \in \mathcal{P}$ *where the above numbers are attained.*

Proof. By Fact 1, there is a woman who receives exactly one proposal. WLOG, say w_n is one such woman. The other women $w_1, \ldots, w_{n-1}$ can receive up to a maximum of n proposals, one from each man. This suggests an upper bound of $n(n-1) + 1 = n^2 - n + 1$ on the number of proposals in men-proposing DA.

Moreover, all men make proposals in the first round, so the first round must consist of n proposals, whereas all the remaining rounds must have at least one proposal. Together with the above upper bound on the number of proposals, this implies an upper bound of $(n^2 - n + 1) - n + 1 = n^2 - 2n + 2$ on the number of rounds.

In the following preference profile, these upper bounds are achieved.

- For $i \in \{1, \ldots, n-1\}$, m_i has preference $w_i \succ w_{i+1} \succ \cdots \succ w_{n-1} \succ w_1 \succ w_2 \succ \cdots \succ w_{i-1} \succ w_n$.
- m_n has preference $w_1 \succ w_2 \succ \cdots \succ w_n$.
- For $j \in \{1, \ldots, n\}$, w_j has preference $m_{j+1} \succ m_{j+2} \succ \cdots \succ m_n \succ m_1 \succ m_2 \succ \cdots \succ m_j$.

With the above preferences, w_n gets exactly one proposal, and all the men $m_1, \ldots, m_n$ cycle through women $w_1, \ldots, w_{n-1}$ one by one until the final assignment of $m_i \leftrightarrow w_{i-1}, i = 2, \ldots, n$ and $m_n \leftrightarrow w_n$. As argued while getting the expressions of the upper bounds on the number of proposals and rounds, this structure is where the $(n-1)$ women except w_n receive n proposals each and w_n receives only one proposal. Also, this structure has n proposals in the first round and each subsequent round has exactly one proposal made. This is the recipe for getting $n^2 - n + 1$ proposals and $n^2 - 2n + 2$ rounds. So, clearly this profile achieves the upper bound. □

These results prompt us to define the following two classes of preferences.

4.1 MaxProposals and MaxRounds

These two classes of preferences are defined as follows.

Definition 6 (MaxProp and MaxRou). In a DA algorithm, a preference profile $\succ$ satisfies

1. MaxProp, if the proposers make $n^2 - n + 1$ proposals in DA, and
2. MaxRou, if the proposing process in DA happens for $n^2 - 2n + 2$ rounds.

Note that, the above two classes are critically dependent on the proposer. We will denote the classes where the maximum number of proposals (and rounds) are coming from the men-proposing DA as m-MaxProp (and m-MaxRou) respectively. The women-proposing versions of the classes will be denoted as w-MaxProp and w-MaxRou respectively. In the rest of the paper, WLOG, we will imply the men-proposing versions of MaxProp and MaxRou respectively when we refer to them and prove their properties. The results for the women-proposing versions are idential and are skipped. However, in Section 4.4, we show that the classes m-MaxProp and w-MaxProp are disjoint for $n \geqslant 3$. Interestingly, these two classes partially overlap for $n = 2$, and we discuss it in Section 4.5. Our first result shows the relationship between the classes MaxProp and MaxRou.

Theorem 2. *If a preference profile* $\succ$ *satisfies* MaxRou, *then* $\succ$ *also satisfies* MaxProp.

Proof. WLOG, assume men-proposing DA in this case. Suppose a preference profile $\succ$ satisfies MaxRou. This implies that if we run the men-proposing DA algorithm, it would take $n^2 - 2n + 2$ rounds to terminate. We make the following observations directly from the algorithm.

- The first round involves n proposals as nobody is matched in the first round, i.e., each man makes a proposal.
- Each round (except the last one) must see at least one man getting rejected, else the termination criterion of the algorithm is met, and thus, every round (except the first one) has at least one proposal.

Hence, the total number of proposals in $\succ$ is $\geqslant n + n^2 - 2n + 1 = n^2 - n + 1$. By Fact 2, we know that the number of proposals is at most $n^2 - n + 1$. Hence, the number of proposals in $\succ$ must be $= n^2 - n + 1$. Therefore $\succ$ satisfies MaxProp. □

The converse of the above theorem is not true for $n \geqslant 3$ as the following example shows.

Example 3 (MaxProp but not MaxRou for $n \geqslant 3$)**.** Consider the following preference profile involving four men and four women.

$$\begin{pmatrix} m_1: & w_1 \succ w_2 \succ w_3 \succ w_4 & & w_1: & m_2 \succ m_3 \succ m_4 \succ m_1 \\ m_2: & w_3 \succ w_2 \succ w_1 \succ w_4 & ; & w_2: & m_3 \succ m_4 \succ m_1 \succ m_2 \\ m_3: & w_3 \succ w_1 \succ w_2 \succ w_4 & & w_3: & m_4 \succ m_1 \succ m_2 \succ m_3 \\ m_4: & w_1 \succ w_2 \succ w_3 \succ w_4 & & w_4: & m_1 \succ m_2 \succ m_3 \succ m_4 \end{pmatrix}$$

In this example, two men (m_1 and m_3) get rejected in the first round of DA. Both these men propose in the next round and it is easy to check that the number of proposals for this profile is $n^2 - n + 1 = 13$. However, since there are two proposals in round 2, instead of the minimum of one that we require for MaxRou, this profile does not satisfy MaxRou. □

We now state an important lemma which will be used in the following subsections to prove several properties of MaxProp. The result gives a structure of proposals in MaxProp.

Lemma 1. *WLOG, let w_n be the woman who receives exactly one proposal in men-proposing* DA *on $\succ$. If $\succ \in$* MaxProp*, then all men $m \in M$ propose to all women in $W \setminus \{w_n\}$.*

Proof. Since $\succ \in$ MaxProp, we have $n^2 - n + 1$ proposals. Since w_n receives exactly one proposal, the other $n - 1$ women receive a total of $n^2 - n$ proposals. No woman can receive more than n proposals (since there are n men). Hence, the only way $n - 1$ women can receive $n^2 - n$ proposals is if each woman in $W \setminus \{w_n\}$ receives n proposals. Thus, all $m \in M$ must propose to all $w \in W \setminus \{w_n\}$. □

Notice that, if a woman receives proposals from all men, she is always assigned to her most preferred man according to the men-proposing DA. Hence, the following corollary is immediate from the lemma above.

Corollary 1. *If $\succ \in$* MaxProp*, all women except the one who gets exactly one proposal, get matched with their most preferred men. Formally, if w_n is the woman who gets exactly one proposal, then for all $i \in \{1, \ldots, n-1\}$, $\mu(w_i) \succ_{w_i} m_j$ or $\mu(w_i) = m_j$ for all $j \in [n]$.*

4.2 MaxProp implies USM

In this section, we prove one of the major results of this paper that provides a new sufficient condition of USM.

Theorem 3. *If a preference profile $\succ$ satisfies* MaxProp*, then $\succ$ is in* USM*.*

Proof. Suppose a preference profile $\succ$ satisfies MaxProp. We show that, the output of men-proposed DA algorithm (say μ) is also women-optimal. Then, by Theorem 1, μ would be the unique stable matching.

Let w_n be the woman who receives exactly one proposal. By Corollary 1, all other women are matched with their first preferences.

Suppose, there is another stable matching $\mu' \neq \mu$ on the same profile $\succ$, which is more preferable than μ for women. Then, for all $i \in [n]$, either $\mu'(w_i) \succ_{w_i} \mu(w_i)$ or $\mu'(w_i) = \mu(w_i)$, and for some $j \in [n], \mu'(w_j) \succ_{w_j} \mu(w_j)$.

However, $w_1, w_2, \ldots, w_{n-1}$ are already matched to their first preferences by μ. So, $\mu'(w_i) = \mu(w_i)$ for $i = 1, \ldots, n-1$, and $\mu(w_n)$ has to be the only man remaining who has to be matched to w_n even in μ'. Hence, $\mu = \mu'$, which is a contradiction. Thus, μ is women-optimal, and is the unique stable matching. □

However, the converse of the previous theorem is not true. The following example shows that MaxProp is not necessary for USM. In fact, this example does not satisfy SPC either.

Example 4 (USM but neither MaxProp nor SPC)**.** Consider the following preference profile.

$$\begin{pmatrix} m_1: & w_1 \succ w_3 \succ w_2 & & w_1: & m_2 \succ m_1 \succ m_3 \\ m_2: & w_2 \succ w_1 \succ w_3 & ; & w_2: & m_3 \succ m_1 \succ m_2 \\ m_3: & w_1 \succ w_2 \succ w_3 & & w_3: & m_1 \succ m_2 \succ m_3 \end{pmatrix}$$

Since there is no pair of man and woman (m, w) that prefers each other the highest, it is not SPC. The men-proposing DA takes 6 proposals, while the maximum number of proposals is $3^2 - 3 + 1 = 7$. Hence, this profile does not satisfy MaxProp. However, the men-optimal matching (obtained via men-proposing DA) results in all women receiving their most preferred men, which is women-optimal as well. Therefore, this profile belongs to USM. □

From Theorems 2 and 3, the following corollary is immediate.

Corollary 2. *If a preference profile $\succ$ satisfies* MaxRou*, then $\succ$ is in* USM*.*

4.3 MaxProp is disjoint from SPC for $n \geqslant 3$

In this section, we address the relative positions of the SPC and MaxProp classes within the space of USM. We show that these two classes are disjoint.

Theorem 4. *For $n \geqslant 3$, there does not exist any preference profile $\succ \in \mathcal{P}$ that satisfies both* SPC *and* MaxProp*.*

Proof. Suppose, there exists a preference profile $\succ$ that satisfies both the SPC and MaxProp. By definition of SPC, there exists an ordering of men and women such that

1. man m_i prefers w_i over $w_{i+1}, w_{i+2}, \ldots, w_n$, and

2. woman w_i prefers m_i over $m_{i+1}, m_{i+2}, \ldots, m_n$.

Hence, m_1 will be proposing to only w_1, who will never reject him, as he is her top preference. Thus, m_1 makes only one proposal. Since MaxProp holds, we know there are a total of $n^2 - n + 1$ proposals to be made. Hence, the remaining $n - 1$ men make $n^2 - n$ proposals, which means each man makes $(n^2 - n)/(n - 1) = n$ proposals. Since in the men-proposed deferred acceptance algorithm, no man proposes to the same woman twice, each woman has to receive a proposal from all $(n - 1)$ men, i.e., each woman receives $\geqslant n - 1$ proposals. Thus, there is no woman who receives exactly one proposal, and this contradicts Fact 1. Hence we have the theorem. □

Discussions. This result naturally implies that for $n \geqslant 3$, the classes SPC and MaxRou, NCC and MaxProp, as well as NCC and MaxRou are mutually disjoint (see Figure 1 for an illustration).

4.4 m-MaxProp and w-MaxProp are disjoint for $n \geqslant 3$

In this section, we show that the MaxProp classes generated by men-proposing and women-proposing DA are disjoint when there are at least *three* agents on each side of the market.

Theorem 5. *For $n \geqslant 3$, there does not exist any preference profile $\succ \in \mathcal{P}$ that satisfies both* m-MaxProp *and* w-MaxProp.

Proof. Suppose there exists a preference profile $\succ \in \mathcal{P}$ satisfying both m-MaxProp and w-MaxProp. Consider the men-proposing DA algorithm on $\succ$. Since $\succ$ satisfies m-MaxProp, by Corollary 1, each $w \in W \setminus \{w_n\}$ is matched with her most preferred man, where w_n is the woman receiving exactly one proposal.

Using Theorem 3, we also know that $\succ$ satisfies USM, i.e., men-proposing DA and women-proposing DA arrive at the same matching. Hence, women-proposing DA on $\succ$ yields a matching in which each $w \in W \setminus \{w_n\}$ is matched with her most preferred man, by making only one proposal. The remaining woman w_n can make at most n proposals. Thus, women-proposing DA on $\succ$ can have at most $1 \times (n - 1) + n = 2n - 1$ proposals.

Further, $\succ$ satisfies w-MaxProp, which means women-proposing DA on $\succ$ involves $n^2 - n + 1$ proposals (Fact 2). In order for this to happen on $\succ$, it must hold that $n^2 - n + 1 \leqslant 2n - 1$, or $n^2 - 3n + 2 \leqslant 0$. However, we know that for $n \geqslant 3, n^2 - 3n + 2 > 0$. Hence, we have a contradiction.

Therefore, for $n \geqslant 3$, there is no $\succ \in \mathcal{P}$ satisfying both m-MaxProp and w-MaxProp. □

The space of these classes is shown graphically in Figure 1a.

4.5 The curious case of $n = 2$

When the number of agents in each side is two, the structure of these spaces looks a bit different. The classes MaxProp and MaxRou become identical and so does SPC and NCC. Quite surprisingly, MaxProp becomes a subset of SPC. These results are formally stated in the following theorems.

Theorem 6 (MaxProp = MaxRou)**.** *For $n = 2$, every preference profile $\succ$ satisfying* MaxProp *also satisfies* MaxRou.

Proof. For $n = 2$, the maximum number of rounds is $n^2 - 2n + 2 = 2$ and the maximum number of proposals is $n^2 - n + 1 = 3$. Now, consider a preference profile $\succ$ satisfying MaxProp. DA on that profile will need to make 3 proposals. Since round 1 of DA can make at most 2 proposals (as $n = 2$), at least 2 rounds are required to make 3 proposals, and thus, $\succ$ satisfies MaxRou as well. □

Theorem 7 (SPC = NCC)**.** *For $n = 2$, every preference profile $\succ$ satisfying* SPC *also satisfies* NCC.

Proof. Consider the preference profile $\succ$ which satisfies SPC. Thus, we have an ordering of men and women (WLOG, assume $(m_1, m_2), (w_1, w_2))$ in which m_1 prefers w_1 to w_2 and w_1 prefers m_1 to m_2. NCC requires that if $w_l \succ_{m_1} w_k$ where $l > k$, then it must imply $w_l \succ_{m_2} w_k$. However, we note that there is no $l > k$ with $w_l \succ_{m_1} w_k$, hence condition 1 is vacuously true. It is easy to see that the same is true even for condition 2. Hence, whatever be the preference of m_2 and w_2, the NCC conditions are always satisfied. This completes the proof. □

Theorem 8 (MaxProp $\subset$ SPC)**.** *For $n = 2$, every preference profile $\succ$ satisfying* MaxProp *also satisfies* SPC.

Proof. Note that for $n = 2$, if both men has the same top women in their preference list, then it is sufficient to claim that the profile is SPC. This is because, the woman (say w_1) who is this top choice of both the men has exactly one man as her top choice (say m_1). Then it is easy to see that the order $(m_1, m_2), (w_1, w_2)$ is the SPC satisfying order.

Now, let a profile $\succ$ satisfy MaxProp. For $n = 2$, it implies that the men should make $2^2 - 2 + 1 = 3$ proposals. If their top preferences were different women, then DA would complete in round 1 with 2 proposals. Hence, it is necessary to have the same woman as the top preference of both men for $\succ$ to be in MaxProp. With our previous observation, we conclude that this implies that $\succ$ also satisfies SPC. □

The converse of the above result is not true. Indeed, MaxProp is a strict subset of SPC as the following example shows.

Example 5 (SPC but not MaxProp for $n = 2$)**.** Consider the following preference profile.

$$\begin{pmatrix} m_1: & w_1 \succ w_2 \\ m_2: & w_2 \succ w_1 \end{pmatrix}; \begin{pmatrix} w_1: & m_1 \succ m_2 \\ w_2: & m_2 \succ m_1 \end{pmatrix}$$

It is easy to see that SPC is satisfied on this profile with the order being $(m_1, m_2), (w_1, w_2)$. However, the number of proposals in men-proposing DA is 2 while MaxProp requires this to be $2^2 - 2 + 1 = 3$. Hence, this profile does not satisfy MaxProp. □

It is also known that for $n = 2$, SPC also becomes necessary for USM, which is shown by Eeckhout (2000). Here we provide a direct proof of this result.

Theorem 9. *For $n = 2$, a preference profile $\succ$ satisfies* SPC *if and only if it is in* USM.

Proof. Note that the 'only if' direction comes directly from Eeckhout (2000), since the proof holds even for $n = 2$. Hence, we only show the 'if' direction of this result.

We will show that if a profile $\succ$ does not satisfy SPC then it cannot belong to USM. Note that, for SPC to be violated, it is necessary that there does not exist a pair of man and woman who rank each other as their first preference. To make this happen, for $n = 2$, both men cannot have the same woman as their first preference, and both women should also have the man who *does not* rank her at the top as her first preference. Hence, the only two possible preference profiles are

$$\begin{pmatrix} m_1: & w_1 \succ w_2 \\ m_2: & w_2 \succ w_1 \end{pmatrix}; \begin{pmatrix} w_1: & m_2 \succ m_1 \\ w_2: & m_1 \succ m_2 \end{pmatrix} \text{ or } \begin{pmatrix} m_1: & w_2 \succ w_1 \\ m_2: & w_1 \succ w_2 \end{pmatrix}; \begin{pmatrix} w_1: & m_1 \succ m_2 \\ w_2: & m_2 \succ m_1 \end{pmatrix}.$$

In both the profiles, the men-optimal DA yields a different matching that the women-optimal DA. Hence, this profile does not belong to USM. This concludes the proof. □

Eeckhout (2000) claims that SPC is necessary for USM even for $n = 3$, which is not true since we show in Example 4 that there are profiles that are not SPC but admit a unique stable matching.

Relative structures of m-MaxProp and w-MaxProp. Unlike the $n \geqslant 3$ case, here these two classes overlap partially.

Theorem 10. *For $n = 2$, a preference profile $\succ \in \mathcal{P}$*

1. *satisfies* m-MaxProp *iff both men have the same woman as their top preference, and*
2. *satisfies both* m-MaxProp *and* w-MaxProp *iff in addition to the above condition both women also have the same man as their top preference.*

Proof. Part 1: Consider the 'if' direction. If both men have the same woman as the top preference in $\succ$, then in first round of men-proposing DA, two proposals will be made and one of them will be rejected who will propose in the next round. Since the maximum number of proposals for $n = 2$ is $2^2 - 2 + 1 = 3$, this will lead to m-MaxProp. For the 'only if' direction, suppose the two men do not have the same woman as their top preference. Then the men-proposing DA will get over in one round with two proposals, and hence will not belong to m-MaxProp.

Part 2: Now we know that the m-MaxProp class contains only those profiles where the men have the same woman as their top preference. In addition, if we also need the profile $\succ$ to be w-MaxProp, then using the women-equivalent condition of Part 1, we get that it is equivalent to both women also having the same man as their top preference. Therefore, the necessary and sufficient condition for a preference profile to be both m-MaxProp and w-MaxProp is that both men have the same woman as their top preference and both women also have the same man as their top preference. □

Collecting all these results, the space of these conditions is graphically shown in Figure 1b. In the following section, we investigate the gap between this DA-inspired class MaxProp and the class of all USMs.

5 A Characterization of MaxProp

The DA-inspired class MaxProp makes the region between SPC and USM clearer for $n \geqslant 3$. In this section, we focus on finding the exact additional properties of the preference profiles in USM that reduces it to MaxProp. But first we show a few structural properties of MaxProp.

Lemma 2. *If a preference profile $\succ \in \mathcal{P}$ satisfies* MaxProp*, then there must be a woman $w \in W$ who is the* least preferred *woman for each $m \in M$.*

Proof. We prove this result via contradiction. WLOG, suppose woman w_n is the woman who receives exactly one proposal (by Fact 1) when men-proposing DA is run on $\succ$. Suppose there is a man m_i who does not have w_n as his last preference. Let m_i prefer w_n over some woman w_j, $j \neq n$. Then by Lemma 1 (as $\succ$ satisfies MaxProp), m_i must propose to w_j, and since he prefers w_n over w_j, he must propose to w_n before w_j. But, w_n gets exactly one proposal and never rejects the man that proposes her. So m_i cannot propose to w_j after proposing to w_n, since it requires w_n to reject m_i under DA to make that happen. Hence, we reach a contradiction. □

Note that the above lemma claims existence of a woman who is least preferred by every man if the profile satisfies MaxProp. In the proof, we have identified that woman as the woman who receives exactly one proposal in DA.

Lemma 3. *Suppose, a preference profile $\succ$ satisfies* MaxProp*. WLOG, w_n be the woman who is every man's last preference in $\succ$, and m_n get matched with w_n in men-proposing* DA*. Then for each $i \in \{1, \dots, n-1\}$, w_i's first preference is some m_j ($j \neq n$), and m_j's penultimate preference is w_i.*

Proof. From Lemma 2, we know that the woman w_n who is every man's last preference in $\succ$ also receives exactly one proposal in men-proposing DA. By Lemma 1 (as $\succ$ satisfies MaxProp), each woman $w_i \in W \setminus \{w_n\}$ gets proposed by every man in M. This implies that she finally gets matched with her most preferred man. Since m_n gets matched with w_n, w_i's first preference must be some m_j ($j \neq n$).

Again using Lemma 1, m_j proposes to all $(n-1)$ women in $W \setminus \{w_n\}$, and he makes his last proposal to the woman who is finally matched with him, i.e., w_i. Since, m_j's least preferred woman is w_n, w_i must be m_j's penultimate preference in $\succ$. □

Using these results, we will now state a set of conditions that are necessary and sufficient for MaxProp. These conditions also identify the additional structure needed for a preference profile in USM to satisfy MaxProp.

Theorem 11. *A preference profile $\succ$ satisfies* MaxProp (m-MaxProp, *WLOG) if and only if there exists an ordering $m_1, \ldots, m_n$ of M and an ordering $w_1, \ldots, w_n$ of W satisfying the following three conditions:*

1. *w_n is the least preferred woman for each $m_i \in M, i = 1, \ldots, n$.*

2. *For each $i \in \{1, \ldots, n-1\}$, w_i's first preference is m_i, and m_i's penultimate preference is w_i.*

3. *For each $k \in \{1, \ldots, n-1\}$, the second preference of w_k is from $\{m_{k+1}, m_{k+2}, \ldots, m_n\}$.*

Before proving the theorem, we make the following observation. We denote the second preference of woman w_ℓ with $s(w_\ell)$. Let G be the digraph with vertices $\{1, 2, \ldots, n-1\}$ where i is joined to j if and only if $s(w_i) = m_j$. Observe that condition 3 says that the given ordering is a topological ordering of G. We know that a directed graph has a topological ordering if and only if it is acyclic. So, an equivalent interpretation of condition 3 is that G is acyclic.

Proof. ($\Rightarrow$): Consider a preference profile $\succ$ that satisfies MaxProp. Since $\succ$ satisfies MaxProp, conditions 1 and 2 of this theorem follow from Lemmas 2 and 3 respectively. We will prove condition 3 by showing that the digraph G is acyclic. Suppose not. Then, G must have at least one directed cycle C involving at least two vertices. Denote the set of vertices in this cycle as $V(C)$. We will show that there exist two different stable matchings, which contradicts that $\succ$ satisfies MaxProp (since MaxProp implies USM by Theorem 3). Construct a matching μ' as follows. For each edge $i, j \in V(C)$ such that a directed edge exists from i to j in G, $\mu'(w_i) = m_j$. For all the remaining women w_i, where $i \in N \setminus V(C)$, $\mu'(w_i) = m_i$. Note that μ' is a stable matching, because of the following reasons.

- None of women w_i, where $i \in C$ can form a blocking pair. The only better match the woman w_i can get is to be matched with her first preference m_i (since she is currently matched to her second preference and condition 2 says that her top preference is m_i). But that man m_i has w_i as the penultimate preference (condition 2) and w_n as the last preference (condition 1), and is currently matched with none of them under μ'. So, m_i does not find this a profitable deviation.

- The remaining women $w_i, i \in N \setminus V(C)$ cannot form blocking pairs either, since $\mu'(w_i) = m_i$, i.e., they have been matched with their most preferred men (condition 2), with the exception of w_n, who cannot form a blocking pair as she is every man's last preference (condition 1).

However, $\mu(w_i) = m_i$ is also a stable matching, as each w_i gets matched with her most preferred man m_i (except w_n who cannot form a blocking pair due to condition 1). Clearly, $\mu \neq \mu'$, since in μ', at least two women between $1, \ldots, (n-1)$ are matched with their second most preferred men. Thus, we have found two distinct stable matchings for $\succ$ and we have a contradiction to USM (and therefore MaxProp).

($\Leftarrow$): Consider a preference profile $\succ$ satisfying the three conditions of this theorem. Pick any stable matching μ on $\succ$.

First, note that $\mu(w_n) = m_n$, i.e., w_n has to be matched with m_n in every stable matching on $\succ$. This is because if w_n is matched with $m_i \in M \setminus \{m_n\}$ then (m_i, w_i) forms a blocking pair: m_i's least preferred woman is w_n (condition 1) and w_i's most preferred man is m_i (condition 2).

We will prove that it must be $\mu(w_i) = m_i$. Suppose not. Let k be largest such that $\mu(w_k) \neq m_k$. This implies that for all $i \in \{k+1, k+2, \ldots, n\}$, we have $\mu(w_i) = m_i$. Therefore, w_k is matched with neither (a) her first nor (b) her second preference. This is because, (a) condition 2 says that m_k is w_k's most preferred man, and (b) the second preference of w_k i.e. $s(w_k)$ is from $\{m_{k+1}, m_{k+2}, \ldots, m_n\}$ (by condition 3) but they are matched with $\{w_{k+1}, w_{k+2}, \ldots, w_n\}$ respectively (by assumption that k is the largest). But then, w_k can form a blocking pair with $m' := s(w_k)$ that is her second preference, as m' has been matched with his least or penultimate preferences, and would prefer w_k over $\mu(m')$, and we reach a contradiction.

Thus $\mu(m_i) = w_i, \forall i \in M$, is *the* unique stable matching for $\succ$, and hence the men-proposed DA algorithm must arrive at this matching. According to this algorithm, each man m_i starts with proposing to his most preferred woman and proposes to the next woman in his preference profile every time he gets rejected, until he reaches his penultimate woman w_i (except for m_n, who proposes until he reaches his last preference w_n). Each m_i for $i \in \{1, \ldots, n-1\}$ proposes $(n-1)$ times, and m_n proposes n times, adding up to a total of $(n-1)(n-1) + n = n^2 - n + 1$ proposals. Thus, the preference profile $\succ$ satisfies MaxProp.

This concludes both directions of the proof. □

Discussion. Theorem 11 gives the necessary and sufficient conditions of MaxProp in the form of three conditions. It is worth asking how critical each of the conditions is. The following set of examples shows that each of the conditions is tight.

Example 6 (Profile $\succ$ violates condition 1 but satisfies conditions 2 and 3)**.** Consider the following preference profile $\succ$ for $n = 3$.

$$\begin{pmatrix} m_1: & w_3 \succ w_1 \succ w_2 & & w_1: & m_1 \succ m_2 \succ m_3 \\ m_2: & w_1 \succ w_2 \succ w_3 & ; & w_2: & m_2 \succ m_3 \succ m_1 \\ m_3: & w_1 \succ w_2 \succ w_3 & & w_3: & m_1 \succ m_2 \succ m_3 \end{pmatrix}$$

Observe that $\succ$ satisfies conditions 2 and 3 with $\sigma = (2, 1)$, but it violates condition 1, as m_1's least preferred woman is not w_3. Men-proposed DA on $\succ$ yields the matching $\mu = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}$, which requires only 4 proposals. If $\succ$ satisfied m-MaxProp, it would require $3^2 - 3 + 1 = 7$ proposals. Thus, $\succ$ violates m-MaxProp. □

Example 7 (Profile $\succ$ violates condition 2 but satisfies conditions 1 and 3)**.** Consider the following preference profile $\succ$ for $n = 3$.

$$\begin{pmatrix} m_1: & w_1 \succ w_2 \succ w_3 & & w_1: & m_1 \succ m_2 \succ m_3 \\ m_2: & w_2 \succ w_1 \succ w_3 & ; & w_2: & m_2 \succ m_3 \succ m_1 \\ m_3: & w_1 \succ w_2 \succ w_3 & & w_3: & m_1 \succ m_2 \succ m_3 \end{pmatrix}$$

Observe that $\succ$ satisfies conditions 1 and 3 with $\sigma = (2, 1)$, but it violates condition 2, as m_1 and m_2 do not have w_1 and w_2 respectively as their penultimate preferences. Men-proposed

DA on $\succ$ yields the matching $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$, which requires only 5 proposals. If $\succ$ satisfied m-MaxProp, it would require $3^2 - 3 + 1 = 7$ proposals. Thus, $\succ$ violates m-MaxProp. $\square$

Example 8 (Profile $\succ$ violates condition 3 but satisfies conditions 1 and 2)**.** Consider the following preference profile $\succ$ for $n = 3$.

$$\begin{pmatrix} m_1: & w_2 \succ w_1 \succ w_3 & & w_1: & m_1 \succ m_2 \succ m_3 \\ m_2: & w_1 \succ w_2 \succ w_3 & ; & w_2: & m_2 \succ m_1 \succ m_3 \\ m_3: & w_1 \succ w_2 \succ w_3 & & w_3: & m_1 \succ m_2 \succ m_3 \end{pmatrix}$$

Observe that $\succ$ satisfies conditions 1 and 2, but it violates condition 3, as there is no woman $w_{\sigma(1)}$ with m_3 as her second most preferred man. Men-proposed DA on $\succ$ yields the matching $\mu = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$, which requires only 5 proposals. If $\succ$ satisfied m-MaxProp, it would require $3^2 - 3 + 1 = 7$ proposals. Thus, $\succ$ violates m-MaxProp. $\square$

To characterize the distinction between the preference profiles that are MaxProp and MaxRou, we provide the following result that characterizes MaxRou using one additional structural property. Note that if at any intermediate stage of the men-proposing Gale-Shapley algorithm, k men propose, it can lead to at most k rejections. Hence, the following observation is immediate.

Observation 1. *If there are k men who propose in a particular round, then at most k men (not necessarily the same men) can propose in all subsequent rounds.*

Theorem 12. *A preference profile $\succ$ satisfies* MaxRou *if and only if it satisfies the following conditions*

1. *$\succ$ satisfies* MaxProp*, and there exists a woman w_n who is the least preferred woman of each man, and*
2. *each woman in $W \setminus \{w_n\}$ is a top preference of some man.*

Proof. $(\Rightarrow)$: Since $\succ$ satisfies MaxRou, it is MaxProp (Theorem 2) as well, and from Theorem 11, we know that there exists a woman w_n who is the least preferred woman of each man. Hence condition 1 is necessary.

Also, since $\succ$ satisfies MaxRou, by definition, the number of rounds is $n^2 - 2n + 2$. The first round always have n proposals and since MaxRou $\Rightarrow$ MaxProp, (Theorem 2), the remaining $n^2 - n + 1 - n = n^2 - 2n + 1$ number of proposals has to come in the remaining $n^2 - 2n + 1$ rounds. This implies that each subsequent round must have exactly one proposal. Now, the number of proposals in the second round is equal to the number of men rejected in the first round, which must be 1. Since all the men propose to some woman amongst the first $(n-1)$ women (Theorem 11), we must have that all $(n-1)$ women in $W \setminus \{w_n\}$ must receive at least one proposal (else, more than one man will be rejected in the first round of the DA algorithm). This implies that each of the women in $W \setminus \{w_n\}$ must be a top preference of at least one man, which is precisely condition 2.

$(\Leftarrow)$: Since $\succ$ satisfies MaxProp and every woman in $W \setminus \{w_n\}$ gets a proposal in the first round of the DA algorithm, at most one man can be rejected in that round since exactly one woman gets two proposals. In the second and each subsequent rounds, we can have at most one proposal. This is because, from Observation 1 we know that if we have k proposals in some round, then we can have at most k proposals in all subsequent rounds. Since $\succ$ satisfies MaxProp, to get $n^2 - n + 1$ proposals where the first round makes n proposals and every subsequent round makes at most one proposal, we must have $n^2 - 2n + 2$ rounds (1 round $\times n$ proposals + remaining $n^2 - 2n + 1$ rounds $\times 1$ proposal). Hence $\succ$ satisfies MaxRou. $\square$

Now, a naive way to check if a preference profile $\succ$ satisfies MaxProp (MaxRou) is to run the DA algorithm and check if it achieves the maximum number of proposals (rounds). This would take $\mathcal{O}(n^2)$ time. But, using the characterization of MaxProp (MaxRou), i.e., Theorem 11 (Theorem 12), we can do much better. Define the following decision problems isMaxProp($\succ$) and isMaxRou($\succ$) as the problem to determine if $\succ$ satisfies MaxProp and MaxRou respectively.

Corollary 3. *For any preference profile* $\succ$

1. isMaxProp($\succ$) *can be checked in* $\mathcal{O}(n)$*.*
2. isMaxRou($\succ$) *can be checked in* $\mathcal{O}(n)$*.*

Proof. Clearly, condition 1 and 2 of Theorem 11 can be checked in $\mathcal{O}(n)$. Now, we know that whether a directed graph $G(V, E)$ is acyclic can be checked in $\mathcal{O}(|V| + |E|)$. But, the graph G as in Theorem 11 has $n - 1$ vertices and at most $n - 2$ edges. Thus, condition 3 can also be checked in $\mathcal{O}(n)$. Hence, whether $\succ$ satisfies MaxProp can be checked in $\mathcal{O}(n)$.

Further, condition 2 of Theorem 12 can also be checked in $\mathcal{O}(n)$. Hence, whether $\succ$ satisfies MaxRou can be checked in $\mathcal{O}(n)$. □

Discussion. Note that these results also help us in the understanding MaxProp and MaxRou conditions (and thereby USM) in a better way. (i) From the structure given by Theorem 11 (or Theorem 12), it is possible to count what fraction of preference profiles satisfy MaxProp (or MaxRou). (ii) The structures look only at partial preferences. The result says we need to know only the top *two* preferred alternatives of one side (say women), the bottom *two* (top one and bottom *two*, for MaxRou) preferred alternatives of the other side (say men), and does not care about the preferences at the other positions. Therefore, we can apply this result on domains with partial preferences as long as the preferences at these positions are known. (ii) From a practical viewpoint, depending on the applications, such profiles may show up in practice.

6 Conclusions and Future Work

In this paper, we have considered the USM problem from a Gale and Shapley (1962) *deferred acceptance algorithmic* perspective. The properties like MaxProp and MaxRou that counts the number of proposals and rounds respectively in this algorithm yields novel insights into the structure of USM. The takeaway point from this kind of sufficiency condition is its simplicity. Both the MaxProp and MaxRou properties are extremely easy to verify on a given profile (given the characterization result of Theorem 11), and also the existence of USM is easy to check since men and women proposing DA arrives at the same stable matching iff it is USM. In addition to the computational simplicity to sufficiency, these conditions carve out a different and unexplored sub-space of USM (see Figure 1). The variation of these spaces for $n = 2$ and $n \geqslant 3$ is interesting. We also provide the structure of the preference profile that differentiates MaxProp class with USM.

As a future plan, we would like to see if any algorithmic property (of not necessarily DA) can explain the whole of the USM class and if there exists an efficient (better than DA) algorithm that can identify USM.

References

José Alcalde. Exchange-proofness or divorce-proofness? stability in one-sided matching markets. *Economic design*, 1:275–287, 1994. (Cited on page 3)

Michel Balinski and Guillaume Ratier. Of stable marriages and graphs, and strategy and polytopes. *SIAM review*, 39(4):575–604, 1997. (Cited on page 3)

Abhijit Banerjee, Esther Duflo, Maitreesh Ghatak, and Jeanne Lafortune. Marry for what? caste and mate selection in modern india. *American Economic Journal: Microeconomics*, 5(2):33–72, 2013. (Cited on page 2)

Angelina Brilliantova and Hadi Hosseini. Fair stable matching meets correlated preferences. In *Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems*, pages 190–198, 2022. (Cited on page 2)

Kim-Sau Chung. On the existence of stable roommate matchings. *Games and economic behavior*, 33(2):206–230, 2000. (Cited on page 3)

Simon Clark. The uniqueness of stable matchings. *Contributions in Theoretical Economics*, 6(1), 2006. (Cited on pages 1, 3, 5, and 6)

Vincent P Crawford and Elsie Marie Knoer. Job matching with heterogeneous firms and workers. *Econometrica: Journal of the Econometric Society*, pages 437–450, 1981. (Cited on page 1)

Jan Eeckhout. On the uniqueness of stable marriage matchings. *Economics Letters*, 69(1):1–8, 2000. (Cited on pages 1, 2, 3, 5, and 11)

Lars Ehlers and Jordi Massó. Incomplete information and singleton cores in matching markets. *Journal of Economic Theory*, 136(1):587–600, 2007. (Cited on page 1)

David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962. (Cited on pages 1, 4, and 16)

David Gale and Marilda Sotomayor. Ms. machiavelli and the stable matching problem. *The American Mathematical Monthly*, 92(4):261–268, 1985. (Cited on page 1)

David Gale and Marilda Sotomayor. Some remarks on the stable matching problem. *Discrete Applied Mathematics*, 11(3):223–232, 1985. (Cited on pages 2 and 4)

Dan Gusfield and Robert W Irving. Parametric stable marriage and minimum cuts. *Information Processing Letters*, 30(5):255–259, 1989. (Cited on page 5)

Gregory Z Gutin, Philip R Neary, and Anders Yeo. Unique stable matchings. *arXiv preprint arXiv:2106.12977*, 2021. (Cited on pages 3 and 5)

Günter J Hitsch, Ali Hortaçsu, and Dan Ariely. Matching and sorting in online dating. *American Economic Review*, 100(1):130–163, 2010. (Cited on page 2)

Ron Holzman and Dov Samet. Matching of like rank and the size of the core in the marriage problem. *Games and Economic Behavior*, 88:277–285, 2014. (Cited on page 3)

Robert W Irving and Paul Leather. The complexity of counting stable marriages. *SIAM Journal on Computing*, 15(3):655–667, 1986. (Cited on pages 1 and 3)

Kazuo Iwama and Shuichi Miyazaki. A survey of the stable marriage problem and its variants. In *International conference on informatics education and research for knowledge-circulating society (ICKS 2008)*, pages 131–136. IEEE, 2008. (Cited on page 1)

Kazuo Iwama, David Manlove, Shuichi Miyazaki, and Yasufumi Morita. Stable marriage with incomplete lists and ties. In *Automata, Languages and Programming: 26th International Colloquium, ICALP'99 Prague, Czech Republic, July 11–15, 1999 Proceedings*, pages 443–452. Springer, 2002. (Cited on page 1)

Anna R Karlin, Shayan Oveis Gharan, and Robbie Weber. A simply exponential upper bound on the maximum number of stable matchings. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 920–925, 2018. (Cited on page 1)

Bettina Klaus and Flip Klijn. Procedurally fair and stable matching. *Economic Theory*, 27:431–447, 2006. (Cited on page 2)

Patrick Legros and Andrew Newman. Co-ranking mates: Assortative matching in marriage markets. *Economics Letters*, 106(3):177–179, 2010. (Cited on page 3)

David F Manlove. The structure of stable marriage with indifference. *Discrete Applied Mathematics*, 122(1-3):167–181, 2002. (Cited on page 1)

Muriel Niederle and Leeat Yariv. Decentralized matching with aligned preferences. Technical report, National Bureau of Economic Research, 2009. (Cited on page 3)

Parag A Pathak and Tayfun Sönmez. Leveling the playing field: Sincere and sophisticated players in the boston mechanism. *American Economic Review*, 98(4):1636–1652, 2008. (Cited on page 2)

Boris Pittel. The average number of stable matchings. *SIAM Journal on Discrete Mathematics*, 2(4):530–549, 1989. (Cited on page 1)

Philip J Reny. A simple sufficient condition for a unique and student-efficient stable matching in the college admissions problem. *Economic Theory Bulletin*, 9(1):7–9, 2021. (Cited on pages 3 and 5)

Antonio Romero-Medina and Matteo Triossi. Acyclicity and singleton cores in matching markets. *Economics Letters*, 118(1):237–239, 2013. (Cited on pages 3 and 5)

Alvin E Roth and Elliott Peranson. The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American economic review*, 89(4):748–780, 1999. (Cited on page 2)

Alvin E Roth. Two-sided matching with incomplete information about others' preferences. *Games and Economic Behavior*, 1(2):191–209, 1989. (Cited on page 1)

Alvin E Roth. Deferred acceptance algorithms: History, theory, practice, and open questions. *international Journal of game Theory*, 36:537–569, 2008. (Cited on page 1)

Nikolaos Tziavelis, Ioannis Giannakopoulos, Rune Quist Johansen, Katerina Doka, Nectarios Koziris, and Panagiotis Karras. Fair procedures for fair stable marriage outcomes. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 7269–7276, 2020. (Cited on page 2)